Homework 3 (due 9/Sep/2013)

Try all the exercises. Submit only those marked with an asterisk (*).

1. (*) Let $A_1, A_2, A_3, \ldots$ be events in a probability space. Write the following events in terms of $A_1, A_2, \ldots$ using the usual set operations (union, intersection, complement).

(1) An infinite number of the events A_i occur.
(2) All except finitely many of the events A_i occur.
(3) Exactly k of the events A_i occur.

2. (Feller, I.8.1) Let $A_1, \ldots, A_n$ be events in a probability space (Ω, p) and let $0 \le m \le n$. Let B_m be the event that at least m of the events $A_1, \ldots, A_n$ occur. Mathematically,

$$B_m = \bigcup_{1 \le i_1 < i_2 < \ldots < i_m \le n} (A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_m}).$$

Show that

$$\mathbf{P}(B_m) = S_m - \binom{m}{1} S_{m+1} + \binom{m+1}{2} S_{m+2} - \binom{m+2}{3} S_{m+3} + \ldots$$

where $S_k = \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} \mathbf{P}(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k})$.

3. Recall the problem of matching two shuffled decks of cards, but with n cards in each deck, so that $\Omega_n = S_n \times S_n$ and $p_{(\pi,\sigma)} = \frac{1}{(n!)^2}$ for each $(\pi, \sigma) \in \Omega$. Let A_m be the event that there are exactly m matches between the two decks[1].

(1) For fixed $m \ge 0$, show that $\mathbf{P}(A_m) \to e^{-1} \frac{1}{m!}$ as $n \to \infty$.
(2) Assume that the approximations above are valid for $n = 52$ and $m \le 10$. Find the probability that there are at least 10 matches.

[**Remark:** You may use the result of the previous problem to solve this one].

4. Place r_n distinguishable balls in n distinguishable urns. Let A_n be the event that at least one urn is empty[2].

(1) If $r_n = n^2$, show that $\mathbf{P}(A_n) \to 0$ as $n \to \infty$.
(2) If $r_n = Cn$ for some fixed constant C, show that $\mathbf{P}(A_n) \to 1$ as $n \to \infty$.
(3) Can you find an increasing function $f(\cdot)$ such that if $r_n = f(n)$, then $\mathbf{P}(A_n)$ does not converge to 0 or 1? [**Hint:** First try $r_n = n^{\alpha}$ for some α, not necessarily an integer][3].

5. (*) (submit first two parts only). A box contains N coupons labelled $1, 2, \ldots, N$. Draw m_N coupons at random, with replacement, from the box. Let A_N be the event that every coupon from the box has appeared at least once in the sample.

(1) If $m_N = N^2$, show that $\mathbf{P}(A_N) \to 1$ as $N \to \infty$.
(2) If $m_N = CN$ for some fixed constant C, show that $\mathbf{P}(A_N) \to 0$ as $N \to \infty$

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[1] Strictly speaking, we should write $A_{n,m}$, since the $A_{n,m} \subseteq \Omega_n$ but for ease of notation we omit the subscript n. Similarly, it would be appropriate to write p_n and $\mathbf{P}_n$ for the probabilities, but again, we simplify the notation when there is no risk of confusion.

[2] Similar to the previous comment, here it would be appropriate to write $\mathbf{P}_n(A_n)$ as the probability spaces are changing, but we keep the notation simple and simply write $\mathbf{P}(A_n)$.

[3] The third parts of this question and the next may be challenging. Even if you cannot solve them, try to understand what the problem is asking for.

(3) (Do not submit this part!). Can you find an increasing function $f(\cdot)$ such that if $m_N = f(N)$, then $\mathbf{P}(A_N)$ does not converge to 0 or 1? [**Hint:** See if you can relate this problem to the previous one].

6. (*) A random experiment is described and a random variable observed. In each case, write the probability space, the random variable and the pmf of the random variable.

(1) Two fair dice are thrown. The sum of the two top faces is noted.

(2) Deal thirteen cards from a shuffled deck and count (a) the number of red cards (i.e., diamonds or hearts), (b) the number of kings.

7. (*) Find $\mathbf{E}[X]$ and $\mathbf{E}[X^2]$ for the following random variables.

(1) $X \sim \text{Geo}(p)$.

(2) $X \sim \text{Hypergeo}(N_1, N_2, m)$.

8. Let X be a non-negative integer-valued random variable with CDF $F(\cdot)$. Show that $\mathbf{E}[X] = \sum_{k=0}^{\infty}(1 - F(k))$.

9. A coin has probability p of falling head. Fix an integer $m \geq 1$. Toss the coin till the m^{th} head occurs. Let X be the number of tosses required.

(1) Show that X has pmf

$$f(k) = \binom{k-1}{m-1} p^m (1-p)^{k-m}, \quad k = m, m+1, m+2, \ldots.$$

(2) Find $\mathbf{E}[X]$ and $\mathbf{E}[X^2]$.

[**Note:** When $m = 1$, you should get the Geometric distribution with parameter p. We say that X has *negative-binomial distribution*. Some books define $Y := X - m$ (the number of tails till you get m heads) to be a negative binomial random variable. Then, Y takes values $0, 1, 2, \ldots$.]