Homework 5 (due 22/Oct/2013)

Try all the exercises. Submit only those marked with an asterisk (*).

1. (*) Let A, B be two events in a common probability space. Write the joint distributions (joint pmf) of the following random variables.

1. $X = \mathbf{1}_A$ and $Y = \mathbf{1}_B$.
2. $X = \mathbf{1}_{A\cap B}$ and $Y = \mathbf{1}_{A\cup B}$.

2. Let (X,Y) have the bivariate normal distribution with density

$$f(x,y) = \frac{\sqrt{ab-c^2}}{2\pi} e^{-\frac{1}{2}\left[a(x-\mu)^2+b(y-\nu)^2+2c(x-\mu)(y-\nu)\right]}.$$

Show that the marginal distributions are one-dimensional normal and find the parameters. For what values of the parameters are X and Y independent?

3. Let r balls be placed in m bins at random. Let X_k be the number of balls in the k^{th} bin. Recall that $(X_1,\dots,X_m)$ has a multinomial distribution. Find the joint distribution of (X_1, X_2) and the marginal distribution of X_1 and of X_2.

4. (*) [*Submit only parts (1) and (2)*]

1. Let X and Y be independent integer-valued random variables with pmf f and g respectively. That is, $\mathbf{P}\{X = k\} = f(k)$ and $\mathbf{P}\{Y = k\}g(k)$ for every $k \in \mathbb{Z}$. Then, show that $X+Y$ has the pmf h given by $h(k) = \sum_{n\in\mathbb{Z}} f(n)g(k-n)$ for each $k \in \mathbb{Z}$.
2. Let $X \sim \text{Pois}(\lambda)$ and $Y \sim \text{Pois}(\mu)$ and assume that X and Y are independent. Show that $X+Y \sim \text{Pois}(\lambda+\mu)$.
3. Let $X \sim \text{Bin}(n, p)$ and $Y \sim \text{Bin}(m, p)$ and assume that X and Y are independent. Show that $X+Y \sim \text{Bin}(n+m, p)$.
4. Let $X \sim \text{Geo}(p)$ and $Y \sim \text{Geo}(p)$ and assume that X and Y are independent. Show that $X+Y$ has negative binomial distribution and find the parameters.

5. (*) [*Submit only parts (1) and (2)*]

1. Let X and Y be independent random variables with densities $f(x)$ and $g(y)$ respectively. Use the change of variable formula to show that $X+Y$ has the density $h(u)$ given by $h(u) = \int_{-\infty}^{\infty} f(s)g(u-s)ds$.
2. Let X, Y be independent Unif$[-1,1]$ random variables. Find the density of $X+Y$.
3. Let $X \sim \text{Gamma}(\mu,\lambda)$ and $Y \sim \text{Gamma}(\nu,\lambda)$ and assume that X and Y are independent. Show that $X+Y \sim \text{Gamma}(\mu+\nu,\lambda)$.

4. Let $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$ and assume that X and Y are independent. Show that $X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

6. Find examples of discrete probability spaces and events A, B, C so that the following happen.

1. The events A, B, C are pairwise independent but not mutually independent.
2. $\mathbf{P}(A \cap B \cap C) = \mathbf{P}(A)\mathbf{P}(B)\mathbf{P}(C)$ but A, B, C are not independent.