FINAL EXAMINATION
PROBABILITY AND STATISTICS
4 DEC 2013, 2PM-5PM

Instructions: *The duration of the test is 3 hours. The maximum you can score is* 50*. The marks for each question is indicated in bold,* [**5**] *means that part carries* 5 *marks. Give all details but try to write succinctly. Start each answer on a new page. Clearly state the results you use.*

1. Fix $a > 0$ and $\alpha > 0$. Let X be a random variable with density

$$f(t) = \begin{cases} \alpha a^{\alpha} t^{-\alpha-1} & \text{if } t > a \\ 0 & \text{if } t \leq a. \end{cases}$$

(1) **[5]** Find the expectation and variance of X.

(2) **[5]** A random number generator gives uniform$([0,1])$ random numbers. Explain how to generate random numbers from the density $f(t)$.

2. Two shuffled decks of cards are placed beside each other.

(1) **[4]** Describe the probability space (sample space and elementary probabilities).

(2) **[6]** Let A be the event that there is a match between the two decks within the first thirteen cards. Show that $0.22 \leq \mathbf{P}(A) \leq 0.25$.

3. The two problems below are not related.

(1) **[5]** If $G_1 \sim \text{Geo}(p_1)$ and $G_2 \sim \text{Geo}(p_2)$ are independent, find $\mathbf{P}\{G_1 < G_2\}$ in terms of p_1 and p_2.

(2) **[5]** Find three random variables X, Y, Z such that $\text{Cov}(X,Y) > 0$ and $\text{Cov}(Y,Z) > 0$ but $\text{Cov}(X,Z) < 0$.

4. Let $X_1, \ldots, X_n$ be i.i.d. with uniform distribution on $[\theta - a, \theta + a]$ where θ, a are unknown parameters.

(1) **[5]** Find the MLE $(\hat{\theta}, \hat{a})$ of (θ, a).

(2) **[5]** Is $\hat{\theta}$ an unbiased estimate of θ? Is $\hat{a}$ an unbiased estimate of a? Justify your answers.

5. Let $X_1, \ldots, X_n$ be i.i.d. $N(\mu, \sigma^2)$, where μ, σ^2 are both unknown. Summary statistics of the data obtained in an experiment are given as follows:

$$n = 20, \quad \sum_{i=1}^{n} X_i = 60, \quad \sum_{i=1}^{n} X_i^2 = 240.$$

(1) **[5]** Find a two-sided confidence interval for μ with confidence level 0.90.

(2) **[5]** Find an upper bound for σ^2 with confidence level 0.90.

6. **[10]** A gambling game in a casino consists of drawing one coupon out of a box containing a large number of blue coupons and an equal number of red coupons. If the coupon drawn is blue, the gambler wins. If the coupon drawn is red, the gambler loses.

It is suspected that the casino is cheating by having more red coupons in the box than blue ones. To test this theory, a gambler plays the game 100 times, out of which she wins 37 times and loses 63 times.

Formulate this as a hypothesis testing problem and carry out the test at significance level 0.05. Report the p-value.

Tables of distributions

Normal CDF at z			χ^2**-distribution quantiles**			t**-distribution quantiles**			
z	+0.00	+0.05	n=d.f	$\chi^2_n(0.10)$	$\chi^2_n(0.90)$	n=d.f	$t_n(0.10)$	$t_n(0.05)$	$t_n(0.025)$
0	0.5	0.51994	1	2.7055	0.015791	1	3.0777	6.3138	12.706
0.1	0.53983	0.55962	2	4.6052	0.21072	2	1.8856	2.92	4.3027
0.2	0.57926	0.59871	3	6.2514	0.58437	3	1.6377	2.3534	3.1824
0.3	0.61791	0.63683	4	7.7794	1.0636	4	1.5332	2.1318	2.7764
0.4	0.65542	0.67364	5	9.2364	1.6103	5	1.4759	2.015	2.5706
0.5	0.69146	0.70884	6	10.645	2.2041	6	1.4398	1.9432	2.4469
0.6	0.72575	0.74215	7	12.017	2.8331	7	1.4149	1.8946	2.3646
0.7	0.75804	0.77337	8	13.362	3.4895	8	1.3968	1.8595	2.306
0.8	0.78814	0.80234	9	14.684	4.1682	9	1.383	1.8331	2.2622
0.9	0.81594	0.82894	10	15.987	4.8652	10	1.3722	1.8125	2.2281
1	0.84134	0.85314	11	17.275	5.5778	11	1.3634	1.7959	2.201
1.1	0.86433	0.87493	12	18.549	6.3038	12	1.3562	1.7823	2.1788
1.2	0.88493	0.89435	13	19.812	7.0415	13	1.3502	1.7709	2.1604
1.3	0.9032	0.91149	14	21.064	7.7895	14	1.345	1.7613	2.1448
1.4	0.91924	0.92647	15	22.307	8.5468	15	1.3406	1.7531	2.1314
1.5	0.93319	0.93943	16	23.542	9.3122	16	1.3368	1.7459	2.1199
1.6	0.9452	0.95053	17	24.769	10.085	17	1.3334	1.7396	2.1098
1.7	0.95543	0.95994	18	25.989	10.865	18	1.3304	1.7341	2.1009
1.8	0.96407	0.96784	19	27.204	11.651	19	1.3277	1.7291	2.093
1.9	0.97128	0.97441	20	28.412	12.443	20	1.3253	1.7247	2.086
2	0.97725	0.97982	21	29.615	13.24	21	1.3232	1.7207	2.0796
2.1	0.98214	0.98422	22	30.813	14.041	22	1.3212	1.7171	2.0739
2.2	0.9861	0.98778	23	32.007	14.848	23	1.3195	1.7139	2.0687
2.3	0.98928	0.99061	24	33.196	15.659	24	1.3178	1.7109	2.0639
2.4	0.9918	0.99286	25	34.382	16.473	25	1.3163	1.7081	2.0595
2.5	0.99379	0.99461	26	35.563	17.292	26	1.315	1.7056	2.0555
2.6	0.99534	0.99598	27	36.741	18.114	27	1.3137	1.7033	2.0518
2.7	0.99653	0.99702	28	37.916	18.939	28	1.3125	1.7011	2.0484
2.8	0.99744	0.99781	29	39.087	19.768	29	1.3114	1.6991	2.0452
2.9	0.99813	0.99841	30	40.256	20.599	30	1.3104	1.6973	2.0423