PROB & STAT (HW: 4) [PRACTICE PROBLEM SOLUTIONS]

Q1.

Soln a) p.m.f of $Pois(\lambda)$ is

$$p_k = e^{-\lambda} \frac{\lambda^k}{k!} \; ; \quad k \geq 0$$

Now $p_{k+1} \geq p_k$

$$\iff e^{-\lambda} \frac{\lambda^{k+1}}{(k+1)!} \geq e^{-\lambda} \frac{\lambda^k}{k!}$$

$$\iff \lambda \geq k+1$$

$$\iff k \leq \lambda - 1$$

i.e p_k is increasing for $k \leq \lfloor \lambda - 1 \rfloor$

i.e mode of $Pois(\lambda)$ is

$$p_{\lfloor \lambda - 1 \rfloor + 1} = p_{\lfloor \lambda \rfloor}$$

b) p.m.f of Hypergeometric (M, W, K) distribution

is $$p_k = \frac{\binom{K}{k}\binom{W}{M-k}}{\binom{K+W}{M}}$$

$$p_{k+1} \geq p_k \iff \frac{\binom{K}{k+1}\binom{W}{M-(k+1)}}{\binom{K+W}{M}} \geq \frac{\binom{K}{k}\binom{W}{M-k}}{\binom{K+W}{M}}$$

$$\iff (K-k)(M-k) \geq (k+1)(W-M+(k+1))$$

$$\iff k \leq (KM+M-W-1)/(W+K+2)$$

i.e p_k is increasing for $k \leq \lfloor (KM+M-W-1)/(W+K+2) \rfloor$, i.e. p_{k+1} for $k = \lfloor (KM+M-W-1)/(W+K+2) \rfloor$ is the mode

89.

<u>soln</u>:

Let B_i denote box i.

Let 1 & 0 denote presence & absence of prize in box i respectively.

B_1	B_2	B_3
1	0	0
0	1	0
0	0	1

Let B_i be the box chosen by the player.

Now $\Omega = \{(1,0,0), (0,1,0), (0,0,1)\}$ are possible outcomes

and $\Pr\{\{(1,0,0)\}\} = \Pr\{\{(0,1,0)\}\} = \Pr\{\{(0,0,1)\}\} = 1/3$

$\Pr\{\text{prize is in original choice}\}$

$= \Pr\{\text{prize is in box } B_i\}$

$= 1/3.$

<u>Rk</u>: After 2-stages of the game

given correct choice of box:
- $(1,*,0)$ — appears with probability $\frac{1}{3}\cdot\frac{1}{2} = \frac{1}{6}$
- $(1,0,*)$ — " $\frac{1}{3}\cdot\frac{1}{2} = \frac{1}{6}$

given incorrect choice of box:
- $(0,0,*)$ — " $1/3$
- $(0,*,0)$ — " $1/3$

6Q. Let $U \sim \text{Uniform}[0,1]$. Find the density and distribution fns of

(a) $U^p, (p>0)$ (b) $U/(1-U)$, (c) $\log(1/U)$, (d) $\frac{2}{\pi}\sin^{-1}(U)$

Soln Given $U \sim \text{Uniform}[0,1]$

a). $f_U(t) = \begin{cases} 0 & t>1 \text{ or } t<0 \\ 1 & 0 \le t \le 1 \end{cases}$

i.e. $F_U(t) = \begin{cases} 0 & t<0 \\ t & 0 \le t \le 1 \\ 1 & t>1 \end{cases}$

Now for $t \in (0,1)$, $t^{1/p} \in (0,1)$.

$F_{U^p}(t) = \mathbb{P}\left[U^p \le t\right]$

$= \mathbb{P}\left[U \le t^{1/p}\right]$

i.e $F_{U^p}(t) = F_U(t^{1/p}) = \begin{cases} 0 & t^{1/p} < 0 \\ t^{1/p} & 0 \le t^{1/p} \le 1 \\ 1 & t^{1/p} > 1 \end{cases} = \begin{cases} 0 & t<0 \\ t^{1/p} & 0 \le t \le 1 \\ 1 & t>1 \end{cases}$

$f_{U^p}(t) = \begin{cases} 0 & t>1 \text{ or } t<0 \\ \frac{1}{p}\, t^{(1/p-1)} & 0 \le t \le 1 \end{cases}$ $\left(\because f_{U^p}(t) = \frac{d}{dt} F_{U^p}(t)\right)$

b). $F_{U/(1-U)}(t) = \mathbb{P}\left[U/(1-U) \le t\right] = \mathbb{P}\left[U \le \frac{t}{1+t}\right]$

$= \begin{cases} \frac{t}{1+t} & \left(\because 0 \le \frac{t}{t+1} \le 1 \text{ for } t \in [0,\infty)\right) \\ 0 & t \le 0 \end{cases}$

$\therefore F_{U/(1-U)}(t) = \begin{cases} \frac{t}{1+t}, & t \in [0,\infty) \\ 0 & t \le 0 \end{cases}$

$$f_{U/(1-U)}(t) = \begin{cases} \frac{1}{(t+1)^2} & t > 0 \\ 0 & t \leq 0 \end{cases}$$

c) $$F_{\log(1/U)}(t) = \mathbb{P}\left[\log 1/U \leq t\right]$$
$$= \mathbb{P}\left[1/U \leq e^t\right]$$
$$= \mathbb{P}\left[U \geq e^{-t}\right] = \begin{cases} 1-e^{-t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$f_{\log(1/U)}(t) = \begin{cases} e^{-t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Now $2/\pi \sin^{-1}(U) \in [0,1]$.

d) $$F_{\frac{2}{\pi}\sin^{-1}(U)}(t) = \mathbb{P}\left[\frac{2}{\pi}\sin^{-1}(U) \leq t\right]$$
$$= \mathbb{P}\left[\sin^{-1}(U) \leq \frac{\pi t}{2}\right] = \mathbb{P}\left[U \leq \sin\frac{\pi t}{2}\right]$$
$$= \begin{cases} 0 & t \leq 0 \\ \sin\frac{\pi t}{2} & 0 \leq t \leq 1 \\ 1 & t \geq 1 \end{cases}$$

$$f_{\frac{2}{\pi}\sin^{-1}(U)}(t) = \begin{cases} 0 & t \notin [0,1] \\ \frac{\pi}{2}\cos\frac{\pi}{2}t & t \in (0,1) \end{cases}$$

7. $X \sim N(0,1)$. Find the density of (a) $aX+b$, (where, $a,b \in \mathbb{R}$, $a \neq 0$)

(b) X^2

(c) X^3

(d) e^X

Solⁿ:- Let $Y = aX+b$

C.D.F of Y is $F_Y(y) = P[Y \le y]$.

$$= P[aX+b \le y]$$

$$= P\left(X \le \frac{y-b}{a}\right)$$

$$= F_X\left(\frac{y-b}{a}\right)$$

Now differentiating both side w.r.t "y" we get

$$f_Y(y) = f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a}$$

$$= \frac{1}{a} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-b)^2}{2a^2}}; \quad -\infty < y < \infty$$

b/ $Y = X^2$

then $F_Y(y) = P[Y \le y]$; $y > 0$

$$= P[X^2 \le y]$$

$$= P(-\sqrt{y} \le X \le \sqrt{y})$$

$$= P[X \le \sqrt{y}] - P(X \le -\sqrt{y})$$

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

differentiating both side of the above equation and using chainrule we get $\left[\because f_Y(y) = \frac{d}{dy} F_Y(y)\right]$

$$f_Y(y) = f_X(\sqrt{y}) \cdot \frac{d}{dy}(\sqrt{y}) - f_X(-\sqrt{y}) \cdot \frac{d}{dy}(-\sqrt{y})$$

$$= \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y})$$

$$= \frac{1}{2\sqrt{y}\sqrt{2\pi}} e^{-\frac{y}{2}} + \frac{1}{2\sqrt{y} \cdot \sqrt{2\pi}} e^{-y/2}$$

$$= \frac{y^{-1/2}}{\sqrt{2\pi}} e^{-y/2} \; ; \quad 0 \le y < \infty$$

$$\therefore f_Y(y) = \begin{cases} \frac{y^{-1/2}}{\sqrt{2\pi}} e^{-\frac{y}{2}} & 0 < y < \infty \\ 0 & \text{otherwise.} \end{cases}$$

(c) $Y = X^3$ as X ranges from $-\infty < x < \infty$
$Y = X^3$ ranges from $-\infty$ to ∞.

Now $F_Y(y) = P(Y \le y)$
$= P(X^3 \le y)$
$= P(X \le y^{1/3})$
$= F_X(y^{1/3})$ [differentiating w.r.t. y]

then $f_Y(y) = f_X(y^{1/3}) \cdot \frac{d}{dy}(y^{1/3})$

$$= \frac{y^{-2/3}}{3\sqrt{2\pi}} e^{-\frac{y^{2/3}}{2}} ; \; -\infty < y < \infty$$

d/ $Y = e^X$ as X ranges from $-\infty < x < \infty$
$Y = e^X$ ranges from $0 < y < \infty$

Now $F_Y(y) = P(Y \le y)$
$= P(e^X \le y)$
$= P(X \le \log y)$
$= F_X(\log y)$

then $f_Y(y) = f_X(\log y) \cdot \frac{d}{dx}(\log y)$

$$= \frac{1}{\sqrt{2\pi} \cdot y} e^{-\frac{(\log y)^2}{2}} ; \quad 0 < y < \infty$$

$$\therefore f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi} \cdot y} e^{-\frac{(\log y)^2}{2}} & 0 < y < \infty \\ 0 & \text{otherwise.} \end{cases}$$

8/ If F is a C.D.F, show that it can have atmost countably many discontinuity points.

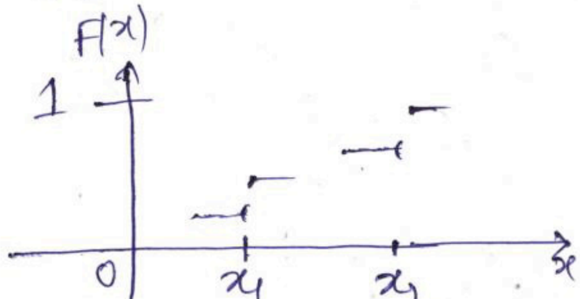
Sol^n:- Since "F" is C.D.F

(i) $0 \le F(x) \le 1 \quad \forall x \in \mathbb{R}$

(ii) F is nondecreasing

(iii) F is Right continuous.

So only point of discontinuity of "F" is Jump discontinuity.



if x_1 is a point of discontinuity of F

then, $F(x_1^-) = \lim\limits_{t \to x_1^-} F(t)$

& $F(x_1^+) = \lim\limits_{t \to x_1^+} F(t)$ both exist and

$F(x_1^-) < F(x_1^+)$.

Since Q (set of rational numbers are dense in $\mathbb{R}$)

we can find out some rational number inbetween $F(x_1^-)$ & $F(x_1^+)$ call it $f(x_1)$

[i.e $F(x_1^-) < f(x_1) < F(x_1^+)$]

if x_2 is another point of discontinuity of F and $x_1 < x_2$

then using nondecreasing property of "F"

we can show that $F(x_1^+) \le F(x_2^-)$

and by construction of "f"

$F(x_1^-) < f(x_1) < F(x_1^+) \le F(x_2^-) < f(x_2) \le F(x_2^+)$

which implies that $f(x_1) \neq f(x_2)$ when ever $x_1 \neq x_2$.

thus $f : \left\{ \begin{array}{l} \text{Set of points} \\ \text{of discontinuity} \\ \text{of } F \end{array} \right\} \longrightarrow \mathbb{Q}$

is one to one

Let $A = \{\text{points of discontinuity of } F\}$

then. $f : A \longrightarrow f(A)$ is a bijection.

Since $f(A) \subset \mathbb{Q}$

$f(A)$ is Countable. [$\because$ Any subset of a Countable set is Countable]

$\Rightarrow$ A is Countable, since "f" is bijection.

4)Q
soln. Given that

X has c.d.f F ; $a > 0$ & $b \in \mathbb{R}$

$Y = aX + b$

The c.d.f of Y

$$F_Y(t) = \mathbb{P}[Y \leq t] = \mathbb{P}[aX + b \leq t] = \mathbb{P}[aX \leq t - b]$$
$$= \mathbb{P}\left[X \leq \frac{t-b}{a}\right]$$
$$= F_X\left(\frac{t-b}{a}\right)$$

Assuming X has p.d.f f_X we have

$$f_Y(t) = f_X\left(\frac{t-b}{a}\right) \cdot \frac{1}{a}.$$