PROBABILITY AND STATISTICS - HOMEWORK 2

DUE DATE: 30/AUG/2016 (IN TUTORIALS)

Homework problems are marked blue (but trying all problems is strongly recommended).

1. Let $A_1, \ldots, A_n$ be events in a common probability space. Let B be the event that *at least two* of the A_is occur. Prove that

$$\mathbf{P}(B) = S_2 - 2S_3 + 3S_4 - \ldots + (-1)^m (m-1) S_m$$

where $S_k = \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} \mathbf{P}\{A_{i_1} \cap \ldots \cap A_{i_k}\}$ for $1 \le k \le n$.

Not for submission: More generally, you may show that the probability that at least ℓ of the A_is occur is equal to

$$S_\ell - \binom{\ell}{1} S_{\ell+1} + \binom{\ell+1}{2} S_{\ell+2} - \binom{\ell+2}{3} S_{\ell+3} + \ldots \tag{1}$$

2. Continuing with the notation of the previous problem, assume the formula given in (1) holds. If C is the event that *exactly* ℓ of the events A_is occur, then show that

$$\mathbf{P}(C) = S_\ell - \binom{\ell+1}{\ell} S_{\ell+1} + \binom{\ell+2}{\ell} S_{\ell+2} - \ldots + (-1)^{n-\ell} S_n. \tag{2}$$

[**Hint:** If you want to prove this directly, without using (1), that is also okay.]

3. If r distinguishable balls are placed at random into m labelled bins, write an expression for the probability that each bin contains at least two balls.

4. A deck of cards is dealt to four players (13 cards each). Find the probability that at least one of the players has two or more *aces*.

5. Let p be the probability that in a gathering of 2500 people, there is some day of the year that is not the birthday of anyone in the gathering. Make reasonable assumptions and argue that $0.3 \le p \le 0.4$.

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6. Consider the problem of a psychic guessing the order of a deck of shuffled cards. Assume complete randomness of the guesses. Use the formula in (2) to derive an expression for the probability that the number of guesses is exactly ℓ, for $0 \leq \ell \leq 52$. Use meaningful approximation to these probabilities and give numerical values (to 3 decimal places) of the probabilities for $\ell = 0, 1, 2 \ldots, 6$.

7. Place r_m distinguishable balls in m distinguishable bins. Let A_m be the event that at least one bin is empty[1].

(1) If $r_n = m^2$, show that $\mathbf{P}(A_m) \to 0$ as $m \to \infty$.

(2) If $r_m = Cm$ for some fixed constant C, show that $\mathbf{P}(A_m) \to 1$ as $n \to \infty$.

(3) Can you find an increasing function $f(\cdot)$ such that if $r_m = f(m)$, then $\mathbf{P}(A_m)$ does not converge to 0 or 1? [**Hint:** First try $r_m = m^\alpha$ for some α, not necessarily an integer].

[1]Here it would be appropriate to write $\mathbf{P}_m(A_n)$ as the probability spaces are changing, but we keep the notation simple and simply write $\mathbf{P}(A_n)$.