PROBABILITY AND STATISTICS - HOMEWORK 3

DUE DATE: 06/SEP/2016 (IN TUTORIALS)

Homework problems are marked blue (but trying all problems is strongly recommended).

1. Let A, B be events with positive probability in a common probability space. We have seen in class that $\mathbf{P}(A \big| B)$ and $\mathbf{P}(B \big| A)$ are not to be confused.

(1) Show that $\mathbf{P}(A \big| B) = \mathbf{P}(B \big| A)$ if and only if $\mathbf{P}(A) = \mathbf{P}(B)$.

(2) Show that $\mathbf{P}(A \big| B) > \mathbf{P}(A)$ if and only if $\mathbf{P}(B \big| A) > \mathbf{P}(B)$. That is, if occurrence of B makes A more likely than it was before, then the occurrence of A makes B more likely than it was.

2. There are 10 bins and the kth bin contains k black and $11 - k$ white balls. A bin is chosen uniformly at random. Then a ball is chosen uniformly at random from the chosen bin.

(1) Find the conditional probability that the chosen ball is black, given that the kth bin was chosen. Use this to compute the (unconditional) probability that the chosen ball is white.

(2) Given that the chosen ball is black, what is the probability that the kth bin was chosen?

3. A fair die is thrown n times. For $1 \leq k \leq n - 1$, let A_k be the event that the kth throw and the $(k + 1)$st throw yield the same result. Are $A_1, \ldots, A_{n-1}$ independent? Are they pairwise independent?

4. Suppose r distinguishable balls are placed in m labelled bins at random. Each ball has probability p_k of going into the kth bin, where $p_1 + \ldots + p_m = 1$. Let X_k be the number of balls that go into the kth bin.

(1) Find the pmf of X_1.

(2) Find the pmf of the random variable $X_1 + X_2$.

5. Two fair dice are thrown and let X be the total of the two numbers that show up. Find the pmf of X. What is the most likely value of X?

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6. Two dice (not necessarily identical, and not necessarily fair) are thrown and let X be the total of the two numbers that turn up. Can you design the two dice so that X is equally likely to be any of the numbers $2, 3, \ldots, 12$?

7. A coin has probability p of falling head. Assume $0 < p < 1$ and fix an integer $m \geq 1$. Toss the coin till the m^{th} head occurs. Let X be the number of tosses required. Show that X has pmf

$$f(k) = \binom{k-1}{m-1} p^m (1-p)^{k-m}, \quad k = m, m+1, m+2, \ldots.$$

Find the CDF of X.

[**Note:** When $m = 1$, this is the Geometric distribution with parameter p. We say that X has *negative-binomial distribution*. Some books define $Y := X - m$ (the number of tails till you get m heads) to be a negative binomial random variable. Then, Y takes values $0, 1, 2, \ldots$.]

8. A box contains n coupons with one number on each coupon. We do not know the numbers but we know that they are distinct. Coupons are drawn one after another from the box, without replacement (i.e., after choosing a coupon at random, it is not put back into the box before drawing the next coupon). If the kth number drawn is larger than all the previous numbers, what is the chance that it is the largest of the n numbers?