PROBABILITY AND STATISTICS - HOMEWORK 4

DUE DATE: 21/SEP/2016 (IN TUTORIALS)

Homework problems are marked blue (but trying all problems is strongly recommended).

1. What is the mode of the (a) Pois(λ) distribution? (b) Hpergeometric(M, W, K) distribution? (Mode means the point(s) where the pmf (or pdf) attains its maximal value).

2. (1) Let $f(k) = \frac{1}{k(k+1)}$ for integer $k \geq 1$. Show that f is a pmf and find the corresponding CDF.

(2) Let $\alpha > 0$ and set $F(x) = 1 - \frac{1}{x^\alpha}$ for $x \geq 1$ and $F(x) = 0$ for $x < 1$. Show that F is a CDF and find the corresponding density function. (This is known as the *Pareto* distribution).

3. Give explicit description of how you would simulate random variables from the following distributions.

(1) The standard Cauchy distribution with density $f(x) = \frac{1}{\pi(1+x^2)}$ for $x \in \mathbb{R}$.

(2) The Beta$(1/2, 1/2)$ distribution with density $\frac{1}{\pi\sqrt{x(1-x)}}$.

(3) (Do not need to submit this) Draw 100 random numbers from either of these densities (on MATLAB or any other program that gives uniform random numbers) using the above procedure and draw the histograms. Compare the histograms to the plot of the densities.

4. Let X be a random variable with distribution function F. Let $a > 0$ and $b \in \mathbb{R}$ and define $Y = aX + b$.

(1) What is the CDF of Y?

(2) If X has a density f, find the density of Y.

5. (1) Let $X \sim \text{Exp}(\lambda)$. Fix $s, t > 0$ and compute the conditional probability of the event $X > t + s$ given that $X > s$.

(2) Let ν be a positive integer. Show that the CDF of Gamma(ν, λ) distribution is given by

$$F(x) = 1 - e^{-\lambda x} \sum_{k=0}^{\nu-1} \frac{\lambda^k}{k!} x^k.$$

Date: September 13, 2016.

6. Let $U \sim \text{Uniform}[0,1]$. Find the density and distribution functions of (a) U^p (where $p > 0$), (b) $U/(1-U)$, (c) $\log(1/U)$, (d) $\frac{2}{\pi}\arcsin(U)$.

7. Let $X \sim N(0,1)$. Find the density of (a) $aX + b$ (where $a, b \in \mathbb{R}$), (b) X^2 , (c) X^3, (d) e^X.

8. In a game, there are three closed boxes, in exactly one of which there is a prize. The player is asked to pick one of the three boxes. The organizer (who knows where the prize is), opens one of the other two boxes and shows that it is empty. Now the player has two choices, can stick to her first choice or to switch to the other closed closed. What should she do? This is known as the *onty hall paradox*. The word "paradox" is used to convey the strong feeling that many have that the probabilities of the two boxes are $1/2$ and $1/2$, since there is always one empty box out of the other two and it gives no information.

To make the problem well-defined, one has to specify how the organizer chooses the empty box. If the player's first choice is empty, then exactly one of the other two boxes is empty and the organizer has no choice but to show that. If the player's first choice is correct, *assume* that the organizer (secretly) tosses a fair coin to choose which of the other two boxes to show. With this specification, show that the probability that the prize is in the original choice is $1/3$ (in other words, if you switch, the chances are higher, that is $2/3$).

9. If F is a CDF, show that it can have at most countably many discontinuity points.