PROBABILITY AND STATISTICS - HOMEWORK 5

DUE DATE: 11/OCT/2016 (IN TUTORIALS)

Homework problems are marked blue (but trying all problems is strongly recommended).

1. Let A, B be two events in a common probability space. Write the joint distributions (joint pmf) of the following random variables.

(1) $X = \mathbf{1}_A$ and $Y = \mathbf{1}_B$.

(2) $X = \mathbf{1}_{A\cap B}$ and $Y = \mathbf{1}_{A\cup B}$.

2. (1) Let $X \sim \text{Exp}(\lambda)$. For any $t, s > 0$, show that $\mathbf{P}\{X > t + s \mid X > t\} = \mathbf{P}\{X > s\}$. (This is called the *memoryless property* of the exponential distribution).

(2) Show that if a non-negative random variable Y has memoryless property (i.e., $\mathbf{P}\{Y > t + s \mid Y > t\} = \mathbf{P}\{Y > s\}$ for all $s, t > 0$), then Y must have exponential distribution.

3. (1) Let X and Y be independent integer-valued random variables with pmf f and g respectively. That is, $\mathbf{P}\{X = k\} = f(k)$ and $\mathbf{P}\{Y = k\}g(k)$ for every $k \in \mathbb{Z}$. Then, show that $X + Y$ has the pmf h given by $h(k) = \sum_{n\in\mathbb{Z}} f(n)g(k-n)$ for each $k \in \mathbb{Z}$.

(2) Let $X \sim \text{Pois}(\lambda)$ and $Y \sim \text{Pois}(\mu)$ and assume that X and Y are independent. Show that $X + Y \sim \text{Pois}(\lambda + \mu)$.

4. Continuation of the previous problem.

(1) Let $X \sim \text{Bin}(n, p)$ and $Y \sim \text{Bin}(m, p)$ and assume that X and Y are independent. Show that $X + Y \sim \text{Bin}(n + m, p)$.

(2) Let $X \sim \text{Geo}(p)$ and $Y \sim \text{Geo}(p)$ and assume that X and Y are independent. Show that $X + Y$ has negative binomial distribution and find the parameters.

5. (1) Let X and Y be independent random variables with densities $f(x)$ and $g(y)$ respectively. Use the change of variable formula to show that $X + Y$ has the density $h(u)$ given by $h(u) = \int_{-\infty}^{\infty} f(s)g(u - s)ds$.

(2) Let X, Y be independent $\text{Unif}[-1, 1]$ random variables. Find the density of $X + Y$.

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6. Continuation of the previous problem.

(1) Let $X \sim \text{Gamma}(\mu, \lambda)$ and $Y \sim \text{Gamma}(\nu, \lambda)$ and assume that X and Y are independent. Show that $X + Y \sim \text{Gamma}(\mu + \nu, \lambda)$.

(2) Let $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$ and assume that X and Y are independent. Show that $X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

7. Let (X, Y) have the bivariate normal distribution with density

$$f(x, y) = \frac{\sqrt{ab - c^2}}{2\pi} e^{-\frac{1}{2}\left[ax^2 + by^2 + 2cxy\right]}.$$

Assume that $a > 0, c > 0, ab - c^2 > 0$ so that this is a valid density.

(1) Show that the marginal distributions are one-dimensional normal and find the parameters.

(2) For what values of the parameters are X and Y independent?

8. A few more exercises in change of variable formula.

(1) If X, Y are independent $N(0, 1)$ random variables, show that X/Y has the Cauchy distribution (with density $\frac{1}{\pi(1+x^2)}$).

(2) If $X \sim \text{Gamma}(\alpha, 1)$, $Y \sim \text{Gamma}(\beta, 1)$ are independent, then show that $X + Y$ and $X/(X + Y)$ are independent, $X + Y \sim \text{Gamma}(\alpha + \beta, 1)$ and $X/(X + Y) \sim \text{Beta}(\alpha, \beta)$.

(3) If X, Y are independent $N(0, 1)$ random variables, show that $X^2 + Y^2$ has $\text{Exp}(1/2)$ distribution.