SECOND MID-TERM EXAMINATION
PROBABILITY AND STATISTICS
3 NOV 2012 – 10:30-12:30

Instructions: *The duration of the test is 2 hours. The maximum you can score is* 45*. The marks for each question is indicated in bold,* [**4**] *means that part carries* 4 *marks. Give all details but try to write succinctly.*

1. Choose the correct answer (no justification needed).

(1) [**2.5**] (X,Y) has joint density $f(x,y)$. Let $U = 2X$ and $V = 3Y$. Then the joint density of (U,V) is

$$(a)\ f(2u,3v),\quad (b)\ f\left(\frac{u}{2},\frac{v}{3}\right),\quad (c)\ 6f(2u,3v),\quad (d)\ \frac{1}{6}f\left(\frac{u}{2},\frac{v}{3}\right).$$

(2) [**2.5**] Let X,Y be i.i.d. $\mathrm{Exp}(\lambda)$ random variables. Let $U = \max\{X,Y\}$ and $V = \min\{X,Y\}$. Then,

$$(a)\ U \sim \mathrm{Exp}(2\lambda)\ \ (b)\ U \sim \mathrm{Exp}(\lambda/2)\ \ (c)\ V \sim \mathrm{Exp}(2\lambda)\ \ (d)\ V \sim \mathrm{Exp}(\lambda/2).$$

(3) [**2.5**] Let X,Y be independent normal random variables with zero means and $\mathrm{Var}(X) = 1$ and $\mathrm{Var}(Y) = 3$. Which of the following pairs of random variables are independent?

$$(a)\ X+Y \text{ and } X-Y\ \ (b)\ 3X+Y \text{ and } X-3Y\ \ (c)\ 3X+Y \text{ and } X-Y\ \ (d)\ X+3Y \text{ and } 3X-Y.$$

(4) [**2.5**] Let X be a random variable with Rayleigh density $f(t) = te^{-t^2/2}$ for $t > 0$. The median of X is

$$(a)\ \log 2\ \ (b)\ \sqrt{\log 2}\ \ (c)\ \sqrt{\log 4}\ \ (d)\ \log 4.$$

2. In each question, state the answer (one number or word) and justify your answer.

(1) [**3**] If X and Y are positively correlated then $-X$ and $-Y$ are negatively correlated.
(2) [**4**] For any random variable X, it is true that $\mathbf{E}[X^4] \geq (\mathbf{E}[X])^4$.
(3) [**3**] A and B are two events with positive probability. If $\mathbf{P}(A \mid B) > \mathbf{P}(A)$ then $\mathbf{P}(B \mid A) > \mathbf{P}(B)$.

3. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with $\mathbf{E}[X_1] = \mu$ and $\mathrm{Var}(X_1) = \sigma^2$. Let $\overline{X}_n = \frac{X_1+\ldots+X_n}{n}$ be the sample mean and let $V_n = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2$ be the sample variance.

(1) [**4**] Compute $\mathbf{E}[\overline{X}_n]$ and $\mathbf{E}[\overline{X}_n^2]$.
(2) [**6**] Compute $\mathbf{E}[V_n]$.

4. The two problems below are unrelated.

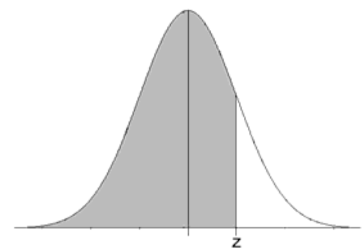
(1) [**4**] If U,V are i.i.d. uniform$([0,1])$ random variables, find the density of $U+V$.
(2) [**6**] Here is the description of a game in a casino. A coupon is drawn from a box containing coupons labelled $1,2,\ldots,20$. If the number is 9 or less then the gambler gets \$1 from the casino. If the number drawn is 10 or more, then the gambler pays \$ 1 to the casino. The gambler play the game 100 times. Using central limit theorem, find an approximation[1] to the probability that the gambler has made a profit.

5. [**10**] A box contains coupons labelled $1,2,\ldots,N$. From this box, draw two coupons without replacement and note the numbers. Call them X_1 and X_2 respectively. Let $Y = (X_1+X_2)/2$. Find the mean and variance of Y.

Date: November 2, 2012.

[1]Normal probability table is given on the next page.

Standard Normal Cumulative Probability Table



Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998