PROBLEM SET 2

DUE ON AUGUST 28 IN TUTORIALS

Submit only those coloured blue. Those in brown may be safely omitted and are meant for the mathematically inclined. Those in black are for practise. Those in white will appear in exams. Write the probability space before calculating any probabilities!

Problem 1. In a group of 5 people, every pair of them could be friends or non-friends. Assume that all possible relationships are equally likely. Find the probability of the event that there are at least three people who are mutually friends. [*Hint:* Don't try anything too clever. The numbers are small, just do proper book-keeping and get the final answer out.]

Problem 2. A drunken man returns home and tries to open the lock of his house from a bunch of n keys by trying them at random till the door opens. Consider two cases: (1) He is so drunk that he may try the same key several times. (2) He is moderately drunk and remembers which keys he has already tried. In both cases, find the probability of the event that he needs k or more attempts to open the door.

Problem 3. Let $A_1, \ldots, A_n$ be events in a common probability space. Let B be the event that *at least two* of the A_is occur. Prove that

$$\mathbf{P}(B) = S_2 - 2S_3 + 3S_4 - \ldots + (-1)^m (m-1) S_m$$

where $S_k = \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} \mathbf{P}\{A_{i_1} \cap \ldots \cap A_{i_k}\}$ for $1 \le k \le n$.

Problem 4. A subset $A \subseteq [0,1]$ is fixed. Two players (who know what is the set A) play a game of constructing a number as follows: The first player writes the first digit after the decimal place (eg., 0.4). Then the second player writes the second digit (eg., 0.42), the first player writes the third digit (eg., 0.423), the second player writes the fourth digit (eg., 0.4230), and so on. In the end, a real number $x = 0.a_1 a_2 \ldots$ is produced. The first player wins if $x \in A$ and the second player wins if $x \notin A$.

(1) If A is the set of rational numbers in $[0,1]$, show that the second player can always win.

(2) (Harder!) If A is an open set (eg., $A = (0.1, 0.2)$), show that the first player can always win or the second player can always win (i.e., one of them has a *winning strategy*).