PROBLEM SET 3

DUE ON SEPTEMBER 11 IN TUTORIALS

Submit only those coloured blue. Those in brown may be safely omitted and are meant for the mathematically inclined. Those in black are for practise. Those in white will appear in exams.

Problem 1. Consider the problem of a "psychic" guessing a deck of 52 cards. For $k = 0, 1, 2, \ldots$, find an exact expression for the probability that the psychic makes exactly k correct guesses. As we did in class for $k = 0$, use the series for e^x to show that the probability is approximately $e^{-1}/k!$. [*Hint:* You could use the result for $k = 0$ that was derived using the inclusion-exclusion principle. Alternately, use the formula given in notes for the probability of exactly m events occurring out of n.]

Problem 2. A box contains one coupon labelled 1, two coupons labelled 2, and so on up to ten coupons labelled 10. Two distinct coupons are drawn at random from the box. Given that the two coupons carry the same label, find the conditional probability that the common label is k (where k is 1 or 2 or ... 10).

Problem 3. Let p be the probability that in a gathering of 2500 people, there is some day of the year that is not the birthday of anyone in the gathering. Make reasonable assumptions and argue that $0.3 \le p \le 0.4$.

Problem 4. In each of the following cases, try to guess whether A and B or independent and if not, whether $\mathbf{P}(B \bigm| A)$ is smaller or larger than $\mathbf{P}(B)$. Then calculate the probabilities and verify the validity of your guesses.

(1) A box contains n coupons labelled $1, 2, \ldots, n$. Coupons are drawn one after another with replacement (a coupon is drawn, the number noted, the coupon is returned to the box, the next coupon is drawn, etc). Let A be the event that the first coupons drawn is an even number. Let B be the event that the second coupon drawn is an even number.

(2) Same as before, except that coupons are drawn without replacement.

(3) Same as first situation (draw with replacement). Let A be the event that the first two coupons drawn are different. Let B be the event that the second and third coupons drawn are different.

(4) A psychic guesses a deck of cards. A is the event that the first guess matches the first card. B is the event that the second guess matches the second card.

(5) 10 balls labelled $1, 2, \dots, 10$ are thrown at random into 4 labelled bins. Let A be the event that the first bin contains a ball with label 4 or less. Let B be the event that the second bin contains a ball labelled 7 or more.

(6) Same situation as the previous one. Let A be the event that the first bin contains a ball with label 4 or less. Let B be the event that the first bin contains a ball labelled 7 or more.

(7) Same situation as the previous one. Let A be the event that the first bin contains a ball with label 4 or less. Let B be the event that the second bin contains a ball labelled 4 or less.

Problem 5. Let n be a large number (in other words, let $n \to \infty$ in the end). Fix two positive integers p and q. A number X is selected at random from $\{1, 2, \dots, n\}$. Let A be the event that X is divisible by p and let B be the event that X is divisible by q. For which p and q is it true that A and B are approximately (as $n \to \infty$) independent? In general, find $\mathbf{P}(A \mid B)$ (limit as $n \to \infty$). For definiteness, first consider the cases $p = 2, q = 3$ and $p = 4, q = 6$. [*Remark*: It is an important principle/heuristic in Number theory that divisibility by distinct primes are independent events.]