PROBLEM SET 6

DUE ON OCTOBER 23 IN TUTORIALS

Submit only those coloured blue. Those in brown may be safely omitted and are meant for the mathematically inclined. Those in black are for practise. Those in white will appear in exams.

Problem 1. From a box of N coupons labelled $1, 2, \dots, N$, we draw three coupons one after another, uniformly at random. Let X_1, X_2, X_3 be the numbers on the coupons. Find the mean and variance of $X_1 + X_2 + X_3$ in the following two situations. (1) The coupons are drawn with replacement. (2) The coupons are drawn without replacement.

Problem 2. Let $X \sim \text{Exp}(\lambda)$.

(1) Find the moments of X. In particular, give the mean and variance.

(2) Find the mean and variance of $\log X$.

[*Remark:* You may get expressions involving Gamma function and its derivatives, they cannot be simplified further. The point to note is that the variance of $\log X$ does not depend on λ. Can you reason why, without doing calculations?]

Problem 3. If X, Y are i.i.d. $N(0,1)$ random variables, show that $X^2 + Y^2$ has $\text{Exp}(1/2)$ distribution.

Problem 4. If X and Y are independent, find the distribution of $X + Y$ in the following situations.

(1) $X \sim \text{Bin}(n, p)$ and $Y \sim \text{Bin}(m, p)$.

(2) $X \sim \text{Pois}(\lambda)$ and $Y \sim \text{Pois}(\lambda')$.

(3) X, Y are i.i.d. $\text{Geo}(p)$.

[*Remark:* Earlier there was a typo in the first part, it said $Y \sim \text{Bin}(m, p')$. There is no simple closed form expression if the second parameters are not the same.]

Problem 5. If X and Y are independent, find the distribution of $Z = T(X, Y)$ in the following situations.

(1) X, Y are i.i.d. $N(0,1)$ and $T(x, y) = x/y$.

(2) X, Y are i.i.d. $\text{Exp}(1)$ and $T(x, y) = X - Y$.

(3) X, Y are i.i.d. Unif$[0,1]$ and $T(x,y) = x + y$.

Problem 6. Let (X, Y) be a bivariate normal with density

$$\frac{\sqrt{ab - c^2}}{2\pi} e^{-\frac{1}{2}(a(x-\mu)^2 + b(y-\nu)^2 + 2c(x-\mu)(y-\nu))}$$

where $a, b, ab - c^2$ are all positive and μ, ν are any real numbers.

(1) Show that $\mathbf{E}[X] = \mu$, $\mathbf{E}[Y] = \nu$, Var$(X) = \sigma_{1,1}$, Var$(Y) \sim \sigma_{2,2}$ and Cov$(X,Y) = \sigma_{1,2}$ where the matrix Σ (called the covariance matrix of (X,Y)) is defined as

$$\Sigma = \begin{bmatrix} \sigma_{1,1} & \sigma_{1,2} \\ \sigma_{2,1} & \sigma_{2,2} \end{bmatrix} := \begin{bmatrix} a & c \\ c & b \end{bmatrix}^{-1}.$$

(2) When are X and Y independent? ("When" means under what conditions on the parameters a, b, c, μ, ν or in terms of $\sigma_{1,1}, \sigma_{2,2}, \sigma_{1,2}, \mu, \nu$?).

(3) If $U = \alpha X + \beta Y$ and $V = \gamma X + \delta Y$, then show that (U, V) has bi-variate normal distribution and find the parameters. Assume that $\alpha\beta - \gamma\delta \neq 0$.

Problem 7. (1) Find a pair of random variables X, Y such that Cov$(X,Y) = 0$ but X and Y are not independent.

(2) If X and Y are Bernoulli random variables (i.e., they take values 0 and 1 only), then show that X and Y independent if and only if they are uncorrelated.

Problem 8. In the following examples, first make an intuitive guess as to whether the variables X and Y are positively correlated or negatively correlated or uncorrelated and then check your answer.

(1) Throw r balls in m bins uniformly at random. Let X and Y be the number of balls in the first and second bins, respectively.

(2) Throw r balls in m bins uniformly at random. Let X be the number of balls in the first bin and let Y be the total number of balls in the first two bins.

(3) U, V are i.i.d. variables with finite variance and $X = U + V$ and $Y = U - V$.

Problem 9. Let U be a random variable and let $X = f(U)$ and $Y = g(U)$ where $f, g : \mathbb{R} \mapsto \mathbb{R}$ are non-decreasing functions. Show that Cov$(X,Y) \geq 0$. [*Remark:* The point is that both go up if U goes up and both go down if U goes down. They ought to be positively correlated!]