PROBLEM SET 7

DUE ON NOVEMBER 6 IN TUTORIALS

Submit only those coloured blue. Those in brown may be safely omitted and are meant for the mathematically inclined. Those in black are for practise. Those in white will appear in exams.

Problem 1. Throw r distinguishable balls into m labelled bins, independently and with probabilities $p_1, \ldots, p_m$ (where $p_1 + \ldots + p_m = 1$). Let X be the number of empty bins. Find the expectation and variance of X.

Problem 2. Let X be a random variable.

(1) Show that $\mathbf{E}[(X-a)^2]$ is minimized when $a = \mathbf{E}[X]$.

(2) Show that $\mathbf{E}[|X-a|]$ is minimized at the medians of X.

[*Hint:* First try the case when X takes finitely many values $t_1 < \ldots < t_n$ with probabilities $1/n$ each.]

Problem 3. Let X be any random variable. Show that $\mathbf{P}\{X \geq t\} \leq e^{-\lambda t}\mathbf{E}[e^{\lambda X}]$ for any $\lambda > 0$. When $X \sim \text{Pois}(1)$, use this to show that $\mathbf{P}\{X \geq k\} \leq e^{-k\log(k/e)-1}$.

Problem 4. A box contains Np coupons where there are p coupons carrying label 1, another p coupons carrying label 2 and so on up to p coupons labelled N. Draw two coupons uniformly at random without replacement and note their labels as X and Y. Find the means and variance of X and of Y and the covariance of X and Y. As $p \to \infty$, what happens to the covariance?

Problem 5. A box contains N coupons labelled $1, 2, \ldots, N$. Coupons are drawn one after another with replacement and the labels noted. After m draws, find the expected number of coupons that have not yet appeared.

(1) If $m = N^2$ and $N \to \infty$, show that this expected number goes to zero.

(2) If $m = CN$ for some constant C, and $N \to \infty$, then show that this expected number goes to infinity.

[*Remark:* How should m change with N so that this expected number converges to a non-zero constant? Work out that the answer is $m = N \log N$.]

Problem 6. Let π be a permutation picked uniformly at random from all permutations. Find the expected number of cycles in π.

Problem 7. Let $X_1, X_2, \ldots$ be i.i.d. random variables taking values ± 1 with equal probability. Show that

$$\mathbf{E}\left[\left(\frac{X_1 + \ldots + X_n}{\sqrt{n}}\right)^p\right] \to \begin{cases} (2k-1) \times (2k-3) \times \ldots \times 1 & \text{if } p = 2k, \\ 0 & \text{if } p = 2k+1. \end{cases}$$

Show that if $Z \sim N(0,1)$, then $\mathbf{E}[Z^p]$ is precisely what is on the right hand side above. This is a sort of justification of the central limit theorem for independent Bernoullis.