Solutions to mid-term exam questions (UM 201)

Problem 1: (1) $P(A_1\cup\ldots\cup A_5) \le P(A_1)+\ldots+P(A_5)$ (union bound)

$\le \frac{1}{10}\times 5 = \frac{1}{2}$ $\boxed{\text{TRUE}}$

$P(A_1\cup\ldots\cup A_5) \ge P(A_1) \ge \frac{1}{20}$

(2) Let $\Omega=\{1,2\}$, $p_1=p_2=\frac{1}{2}$. $A=\{1,2\}$, $B_1=\{1\}$, $B_2=\{2\}$. $\boxed{\text{FALSE}}$

Then $P(A|B_1)=1$ $P(A|B_2)=1$ $P(A|B_1\cup B_2)=1 \ne P(A|B_1)+P(A|B_2)$

[LHS could be more than 1, RHS ≤ 1, hence one realizes it must be false]

(3) $P(D\cap C) = P((A\cup B)\cap C) = P((A\cap C)\cup(B\cap C)) = P[(A\cap C)\cup(B\cap C\cap A^c)]$

disjoint

but $(A\cap C)\cup(B\cap C) = (A\cap C)\cup(B\cap C\cap A^c)$

$= P(A\cap C) + P(B\cap C\cap A^c)$

$= P(A)P(C) + P(B)P(C)P(A^c)$ → independence

$= P(C)[P(A)+P(B\cap A^c)] = P(C)P(A\cup B)$ ∵ $A\cup(B\cap A^c) = A\cup B$.

(4) $X\sim Bin(10,\frac{1}{2})$, $Y\sim Bin(11,\frac{1}{2})$.

$P\{X\ge 5\} = \frac{1}{2^{10}}\left(\binom{10}{5}+\ldots+\binom{10}{10}\right) = \frac{1}{2^{10}}\#\{A\subseteq\{1,2,\ldots,10\} \mid \#A\ge 5\}$ — ①

$P\{Y\ge 5\} = \frac{1}{2^{11}}\left(\binom{11}{5}+\ldots+\binom{11}{11}\right) = \frac{1}{2^{11}}\#\{B\subseteq\{1,2,\ldots,11\} \mid \#B\ge 5\}$ — ②

For each $A\subseteq\{1,2,\ldots,10\}$ associate $A_1 = A$, $A_2 = A\cup\{11\}$ subsets of $\{1,\ldots,11\}$

$\#A\ge 5 \Rightarrow \#A_1\ge 5$ and $\#A_2\ge 5$

Further if $A\ne A'$ then A_1, A_2, A_1', A_2' are distinct.

Hence $\#\{B\subseteq\{1,2,\ldots,11\} \mid \#B\ge 5\} \ge 2\#\{A\subseteq\{1,\ldots,10\} \mid \#A\ge 5\}$

Thus ① $\le$ ②.

This proof shows that if $X\sim Bin(n,\frac{1}{2})$ and $Y\sim Bin(m,\frac{1}{2})$ with $m\ge n$, then $P\{X\ge k\}\le P\{Y\ge k\}$ $\forall k$.

<u>Alternate proof:</u> $\Omega=\{0,1\}^{11}$, $p_\omega = 2^{-11}$ (Toss a fair coin 11 times)

Let $X(\omega)=\omega_1+\ldots+\omega_{10}$ $Y(\omega)=\omega_1+\ldots+\omega_{11}$

Then $X\sim Bin(10,\frac{1}{2})$, $Y\sim Bin(11,\frac{1}{2})$ but $X(\omega)\le Y(\omega)$ $\forall\omega$.

Hence $P\{X\ge k\}\le P\{Y\ge k\}$ as $\{X\ge k\}\subseteq\{Y\ge k\}$.

Problem 2: (1) CDF of Rayleigh density is $F(t) = \int_0^t x e^{-\frac{1}{2}x^2} dx$ (for $t \geq 0$)

Of course $F(t) = 0$ for $t < 0$. $= 1 - e^{-\frac{1}{2}t^2}$ $\left(\because \frac{d}{dx} e^{-\frac{1}{2}x^2} = -x e^{-\frac{1}{2}x^2}\right)$

Thus $F(t) = \frac{1}{2} \Leftrightarrow 1 - e^{-\frac{1}{2}t^2} = \frac{1}{2} \Leftrightarrow e^{-\frac{1}{2}t^2} = \frac{1}{2} \Leftrightarrow -\frac{1}{2}t^2 = \log\frac{1}{2}$
$\Leftrightarrow t = \sqrt{2\log 2}$

As there is a unique t s.t. $F(t) = \frac{1}{2}$ the median is unique and equal to $\sqrt{2\log 2}$

(2) $P\{$bins 1 and 2 are empty and bins 3 and 4 are not empty$\}$

$= \dfrac{2^5 - 2}{4^5}$

$2^5 - 2 \rightarrow$ 2^5 ways to put 5 balls into bins 3 and 4 but subtract the cases where all balls go into one of the two

$4^5 \rightarrow$ 4^5 ways to put 5 balls in 4 bins.

$= \dfrac{30}{2^{10}}$

Similarly, the event that 1 and 3 are empty, 2 and 4 are not empty etc. also have probability $30/2^{10}$. There are $\binom{4}{2} = 6$ such events and they are pairwise disjoint. Hence

$P\{\text{exactly two bins empty}\} = 6 \times \dfrac{30}{2^{10}} = 3 \times \dfrac{15}{2^8} = \dfrac{45}{256}$.

(3) $P\{A \cap B \cap C\} = 1 - P\{(A \cap B \cap C)^c\} = 1 - P\{A^c \cup B^c \cup C^c\}$.

But $P(A^c \cup B^c \cup C^c) \leq P(A^c) + P(B^c) + P(C^c)$ (union bound)
$= (1-0.9) + (1-0.8) + (1-0.7) = 0.6$

Hence $P(A \cap B \cap C) \geq 1 - 0.6 = 0.4$

Equality can be achieved in the union bound if A^c, B^c, C^c are pairwise disjoint. This can be arranged. Hence minimum required is $\underline{0.4}$.

Eg: $\Omega = \{1, 2, \ldots, 10\}$, $p_\omega = \frac{1}{10}$ $\forall \omega \in \Omega$

$A = \{2, 3, \ldots, 10\}$, $B = \{1, 4, 5, \ldots, 10\}$ $C = \{1, 2, 3, 7, 8, 9, 10\}$

$(A^c = \{1\}$, $B^c = \{2, 3\}$, $C^c = \{4, 5, 6\})$.

(4) A_k = event that die throws up k, $1 \leq k \leq 6$. B = event that 4 heads show up.

$P(B|A_k) = \binom{k}{4}\frac{1}{2^k}$ i.e. $= \begin{cases} \binom{6}{4}\frac{1}{2^6} = \frac{15}{2^6} & \text{if } k=6 \\ \binom{5}{4}\frac{1}{2^5} = \frac{10}{2^6} & \text{if } k=5 \\ \binom{4}{4}\frac{1}{2^4} = \frac{4}{2^6} & \text{if } k=4 \end{cases}$ and 0 for $k \leq 3$.

Also $P(A_k) = \frac{1}{6}$ $\forall k$.

By Bayes' rule, $\cancel{P\{A_6|A_6}$ $P\{A_6|B\} = \dfrac{\frac{1}{6}(15/2^6)}{\frac{1}{6}\frac{15}{2^6} + \frac{1}{6}\frac{10}{2^6} + \frac{1}{6}\frac{4}{2^6}} = \dfrac{15}{29}$.

Problem 3: Let the bins be labelled 0 and 1.

$$\Omega = \{01, 001, 0001, \dots\} \cup \{10, 110, 1110, \dots\} \qquad (110 = 1^2 0,\ 1110 = 1^3 0, \dots)$$

with $p_{0^k 1} = p_{1^k 0} = \dfrac{1}{2^{k+1}}$ ($0^k 1$ means the outcome {first k balls in bin 0, $(k+1)^{st}$ ball in bin 1})

$X : \Omega \to \mathbb{R}$ is defined as $X(0^k 1) = k+1$, $X(1^k 0) = k+1$ (# balls thrown)

(1) $P\{X = \ell\} = P\{0^{\ell-1}1, 1^{\ell-1}0\}$ (if $\ell \geq 2$, else $P\{X=\ell\} = 0$)

$$= \frac{1}{2^\ell} + \frac{1}{2^\ell} = \frac{1}{2^{\ell-1}}.$$

Hence the pmf of X is $f(\ell) = \dfrac{1}{2^{\ell-1}}$ for $\ell = 2, 3, 4, \dots$

CDF: Let F be the CDF of X. $F(t) = 0$ if $t < 2$.

For $k \geq 2$, if $k \leq t < k+1$, $F(t) = \displaystyle\sum_{\ell=2}^{k} \frac{1}{2^{\ell-1}} = 1 - \frac{1}{2^{k-1}}$.

(2) $E[X] = \displaystyle\sum_{\ell=2}^{\infty} \ell \frac{1}{2^{\ell-1}} = \frac{1}{(1-\frac{1}{2})^2} - 1$ ($x = 1/2$)

$$= 4 - 1 = 3$$

$$\boxed{\begin{aligned} &\sum_{\ell=2}^{\infty} x^\ell = \frac{1}{1-x} - 1 - x \\ &\downarrow \text{differentiate wrt } x \\ &\sum_{\ell=2}^{\infty} \ell x^{\ell-1} = \frac{1}{(1-x)^2} - 1 \end{aligned}}$$

Median: $F(t) = \begin{cases} 0 & \text{if } t < 2 \\ 1/2 & \text{if } 2 \leq t < 3 \\ 3/4 & \text{if } 3 \leq t < 4 \\ \vdots \end{cases}$ $\Rightarrow P\{X \leq t\} \geq \frac{1}{2}$ if $t \geq 2$

$P\{X \geq t\} = \begin{cases} 1 & \text{if } t \leq 2 \\ 1/2 & \text{if } t \geq 3 \\ 1/4 & \text{if } t \geq 4 \\ \vdots \end{cases}$ $\Rightarrow P\{X \geq t\} \geq \frac{1}{2}$ if $t \leq 3$

Thus all numbers in $[2, 3]$ are medians and these are all the medians.

Problem 4: $X \sim \mathrm{Exp}(1)$. $F_X(t) = 1 - e^{-t}$ if $t \geq 0$.

Point: Discretizing Exp. distribution with equispaced points gives Geometric distribution.

(1) Geo(1/2): pmf $g(k) = \frac{1}{2^k}$, $k = 1, 2, 3, \dots$

$$P\{a \le X < b\} = (1 - e^{-b}) - (1 - e^{-a}) = e^{-a} - e^{-b}.$$

Hence if we take $a = (k-1)\log 2$ $b = k \log 2$, we get $2^{-(k-1)} - 2^{-k} = 2^{-k}$.

Thus if we set $Y = \phi(X)$ where $\phi(x) = \begin{cases} 1 & \text{if } x \in [0, \log 2) \\ 2 & \text{if } x \in [\log 2, 2\log 2) \\ \vdots & \vdots \end{cases}$

$\left(\text{or } \phi(x) = \left\lfloor \frac{x}{\log 2} \right\rfloor + 1\right)$

then $Y \sim$ Geo(1/2).

(2). CDF of logistic distribution is $G(t) = \int_{-\infty}^{t} \frac{e^x}{(1+e^x)^2}\,dx = \frac{-1}{1+e^x}\Big|_{-\infty}^{t} = 1 - \frac{1}{1+e^t}$.

If $X \sim \mathrm{Exp}(1)$, and $\psi : [0, \infty) \to \mathbb{R}$ is strictly increasing and continuous, then $Z = \psi(X)$ has CDF $H(t) = F_X(\psi^{-1}(t)) = 1 - e^{-\psi^{-1}(t)}$

We want $1 - e^{-\psi^{-1}(t)} = 1 - \frac{1}{1+e^t}$ or $e^{-x} = \frac{1}{1+e^{\psi(x)}}$

Hence $1 + e^{\psi(x)} = e^x$ or $\psi(x) = \log(e^x - 1)$.

Thus $Z = \log(e^X - 1) \sim$ logistic.

Problem 5: $P\{X_\lambda \ge n\} = \cancel{1 - \sum_{k=0}^{n-1} e^{-\lambda}\frac{\lambda^k}{k!}} = P\{X_\lambda \ge n-1\} - e^{-\lambda}\frac{\lambda^{n-1}}{(n-1)!}$ — (1)

$$P\{Y_n \le \lambda\} = \int_0^\lambda e^{-t}\frac{t^{n-1}}{(n-1)!}\,dt = \left[-e^{-t}\frac{t^{n-1}}{(n-1)!}\right]_0^\lambda + \int_0^\lambda e^{-t}\frac{t^{n-2}}{(n-2)!}\,dt$$

$$= -e^{-\lambda}\frac{\lambda^{n-1}}{(n-1)!} + P\{Y_{n-1} \le \lambda\} \quad \text{— (2)}$$

if $n \ge 2$

From (1) and (2), it is clear that we only need to show

$P\{X_\lambda \ge 1\} = P\{Y_1 \le \lambda\}$ (then it follows for $n=2$, $n=3$, ... inductively)

But $P\{X_\lambda \ge 1\} = 1 - e^{-\lambda}$ and $Y_1 \sim \mathrm{Exp}(\lambda)$ hence $P\{Y_1 \le \lambda\} = 1 - e^{-\lambda}$.

This completes the proof.