# MA 312 Commutative Algebra / Jan–April 2020

( BS, Int PhD, and PhD Programmes)



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**Lectures :** Tuesday and Thursday ; 15:30–17:00 **Venue:** MA LH-5 / LH-1

## 3. Rings and Modules with Chain Conditions

**Submit a solutions of** *Exercises **ONLY.** **Due Date : Thursday, 13-02-2020**
**Recommended to solve the violet colored** R Exercises

**3.1** Let $k$ be a field.

**(a)** Let $B = k[x]$ be a cyclic $k$-algebra. Then every $k$-subalgebra $A$ of $B$ is a finite type $k$-algebra. (**Hint :** If $f \in A$, $f = \sum_{i=0}^{m} a_i x^i$, $a_m \neq 0$, $m \geq 1$, then $B = \sum_{i=0}^{m-1} k[f]x^i$ is finite over $k[f] \subseteq A$.)

**(b)** Let $B = k[\mathbb{N}^2]$ be the monoid algebra over $k$ of the additive monoid $\mathbb{N}^2$ and let $X := e_{(1,0)}$, $Y := e_{(0,1)}$. Then $B = k[X,Y]$, and the monomials $X^iY^j = e_{(i,j)}$, $(i,j) \in \mathbb{N}^2$, form a $k$-basis of $B$. Let $A$ be the $k$-subalgebra of $B$ generated by the monomials $X^{n+1}Y^n$, $n \in \mathbb{N}$. Then $A$ is not a noetherian ring, much less than a finite type $k$-algebra. (**Hint :** Note that $B$ is the polynomial algebra in two indeterminates $X,Y$ over $k$ and $X^{n+1}Y^n$ does not belong to the ideal (in $A$) generated by $X, \ldots, X^nY^{n-1}$, for every $n \in \mathbb{N}$.)

**(c)** (Ring of integer-valued polynomials) The set

$$\mathrm{Int}(\mathbb{Z}) := \{f \in \mathbb{Q}[X] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\} \subseteq \mathbb{Q}[X]$$

of integer-valued polynomials is obviously a subring of the polynomial ring $\mathbb{Q}[X]$ — called the ring of integer-valued polynomials.[1] For each prime number $p$ and each $\bar{k} \in \mathbb{F}_p$, the subset

$$\mathfrak{M}_{p,\bar{k}} := \{f \in \mathrm{Int}(\mathbb{Z}) \mid v_p(f(\bar{k})) \geq 1\} \subseteq \mathrm{Int}(\mathbb{Z}).$$

is a maximal ideal in $\mathrm{Int}(\mathbb{Z})$ with residue field isomorphic to the prime field $\mathbb{F}_p$. (**Remarks :** One can prove that the maximal spectrum $\mathrm{Spm}\ \mathrm{Int}(\mathbb{Z}) = \{\mathfrak{M}_{p,\bar{k}} \mid p \in \mathbb{P}$ and $\bar{k} \in \mathbb{F}_p\}$. Moreover, one can even describe the prime spectrum $\mathrm{Spec}\ \mathrm{Int}(\mathbb{Z})$ explicitly. Further, none of the non-zero prime ideals in $\mathrm{Int}(\mathbb{Z})$ are finitely generated. )

***3.2** Let $A$ be a ring and let $\mathfrak{a}$ be a non-zero ideal in $A$.

**(a)** If $A$ is a noetherian ring, then every surjective ring endomorphism of $A$ is an automorphism.

**(b)** If $A$ is a noetherian ring, then $A$ and $A/\mathfrak{a}$ are not isomorphic rings.

**(c)** If $A$ is a finite type commutative algebra over the ring $R$, then every surjective $R$-algebra endomorphism $\varphi$ of $A$ is an automorphism. (**Hint :** Suppose that $\varphi(x) = 0$ and $x_1, \ldots, x_m$ ia a $R$-algebra generating system for $A$. The construct a finitely generated $\mathbb{Z}$-subalgebra $R'$ of $R$ such that $R'[x_1, \ldots, x_m]$ contain $x$ as well as $\varphi$ is a surjective endomorphism $R'[x_1, \ldots, x_m]$. — Note that the assertion does not hold for arbitrary commutative algebra. Examples!)

[1] The interest for the ring structure of $\mathrm{Int}\,\mathbb{Z}$ arose only in the last quarter of the 20th century, but it was at least well known at the time of Polya that every integer-valued polynomial $f$ of degree $n$ can be uniquely written as a $\mathbb{Z}$-linear combination : $f(X) = \sum_{k=0}^{n} c_k \mathrm{B}_k(X)$ and the coefficients $c_k$ are recursively given by the formula : $c_k = f(k) - \sum_{i=0}^{k-1} c_i \mathrm{B}_i(k)$. This evokes the Gregory-Newton formula which dates back to the 17th century : $f(X) = \sum_{k=0}^{n} (\Delta^k f)(0)\,\mathrm{B}_k(X)$.

**(d)** If $A$ is a finite type commutative algebra over the ring $R$, then $A$ and $A/\mathfrak{a}$ are not isomorphic as $R$-algebras.

**3.3** Let $(\mathfrak{p}_i)_{i\in\mathbb{N}}$ be a sequence of prime ideals with $\mathfrak{p}_i \not\subseteq \mathfrak{p}_j$ for all $i, j \in \mathbb{N}$ with $i < j$ and put $\mathfrak{a}_n := \cap_{i\leq n}\mathfrak{p}_i$, $n \in \mathbb{N}$. then $(\mathfrak{a}_n)_{n\in\mathbb{N}}$ form a strict descending chain of ideals in $A$. Deduce that if $A$ is artinian, then $\operatorname{Spec} A = \operatorname{Spm} A$ is a finite set, (nil-radical of $A$) $\operatorname{nil} A = \mathfrak{m}_A$ (the Jacobson-radical of $A$) and that $A/\mathfrak{m}_A$ is a finite product of fields. (**Hint:** If $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ are ideals in the ring $A$ and if $\mathfrak{p}$ is a prime ideal in $A$ with $\cap_{i=1}^n \mathfrak{a}_i \subseteq \mathfrak{p}$, then $\mathfrak{a}_i \subseteq \mathfrak{p}$ for some $i \in \{1, \ldots, n\}$. In particular, if $\mathfrak{p} = \cap_{i=1}^n \mathfrak{a}_i$, then $\mathfrak{p} = \mathfrak{a}_i$ for some $i \in \{1, \ldots, n\}$.)

**3.4** Let $A$ be a *commutative ring* and let $V$ be a finite $A$-module.

**(a)** Every surjective endomorphism $f : V \to V$ is bijective.

**(b)** Let $U$ be a submodule $\neq 0$ of $V$. If $V$ noetherian or if $V$ finite, then $V$ and $V/U$ are not isomorphic as $A$-modules. (**Hint:** If they are isomorphic the give a surjective $A$-endomorphism of $V$ with kernel $U$.)

**(c)** Let $W$ an arbitrary $A$-module. If $V \cong V \oplus W$ (as $A$-modules) then $W = 0$. (**Hint:** Use the part (a).)

**3.5** Let $K$ be a field, $I := \mathbb{N} \cup \{\infty\}$, $V := K^{(I)}$ and $e_i$, $i \in I$, be the standard basis of $V$ and $V_n := \sum_{i=0}^n Ke_i$ for $n \in \mathbb{N}$, $V_\infty := \sum_{i\in\mathbb{N}} Ke_i$. The set of $K$-endomorphisms $f$ of $V$ with $f(V_n) \subseteq V_n$ for all $n \in \mathbb{N}$ is a $K$-subalgebra $A$ of $\operatorname{End}_K V$. With respect to the natural $A$-module structure on $V$, besides 0 and $V$, $V_n$, $n \in \mathbb{N}$, and $V_\infty$ are the only $A$-submodules of $V$. The $A$-module $V (= Ae_\infty)$ is cyclic and artinian, but not noetherian.

**3.6** Let $A$ be a commutative ring.

**(a)** Let $V$ be a finite $A$-module and $W$ a noetherian (resp. artinian) $A$-module. Then $\operatorname{Hom}_A(V, W)$ is also noetherian (resp. artinian).

**(b)** Let $V$ be an $A$-module which is noetherian (resp. finite and artinian). Then $\operatorname{End}_A V$ is a noetherian (resp. finite and artinian) $A$-module. In particular, every $A$-subalgebra of $\operatorname{End}_A V$ is noetherian (resp. finite artinian).

***3.7** **(a)** Every artinian module is a direct sum of finitely many indecomposable modules.

**(b)** Every noetherian module is a direct sum of finitely many indecomposable modules. (**Hint:** Suppose not, then $V$ is not decomposable and $V = V_1 \oplus V_2$ with $V_2$ not indecomposable. With this construct an infinite strict decreasing sequence $V_0 \supsetneq V_1 \supsetneq \cdots$ of direct summands in the module $V$ and hence also construct an infinite strict increasing sequence of direct summands.)

**3.8** Let $A$ be a ring and let $V$ be an $A$-module. Suppose that $V = V_1 \oplus \cdots \oplus V_n$ is a direct sum of submodules $V_1, \ldots, V_n$ such that the endomorphism rings $\operatorname{End}_A V_i$ of $V_i$, $1 \leq i \leq n$, are local rings. If $V = W_1 \oplus \cdots \oplus W_m$ is also a direct sum of the *indecomposable* submodules $W_1, \ldots, W_m$, then $m = n$ and there exists a permutation $\sigma \in \mathfrak{S}_n$ with $V_i \cong W_{\sigma(i)}$. (**Hint:** Proof by induction on $n$. Let $P_1, \ldots, P_n$ (resp. $Q_1, \ldots Q_m$) be the families of projections corresponding to the decompositions $V = V_1 \oplus \cdots \oplus V_n$ (resp. $V = W_1 \oplus \cdots \oplus W_m$). For $j \in \{1, \ldots, m\}$, let $P_{1j} := P_1|W_j$ be the restriction of $P_1$ into the image $V_1$ and $Q_{j1} := Q_j|V_1$ be the restriction of $Q_j$ into the image $W_j$. Then $\operatorname{id}_{V_1} = \sum_{j=1}^m P_{1j}Q_{j1}$. Since $\operatorname{End}_A V_1$ is local, there exists $r \in \{1, \ldots, m\}$ such that $P_{1r}Q_{r1}$ is an isomorphism. Now, it follows from the following easy Exercise[2] that $Q_{r1} : V_1 \to W_r$ is an isomorphism.)

[2] **Exercise:** Let $f : V \to W$ and $g : W \to X$ be homomorphisms of modules over a ring. If the composition $gf$ is an isomorphism, then $f$ is injective, $g$ is surjective and $W \xrightarrow{\sim} \operatorname{Img} f \oplus \operatorname{Ker} g$, $w \mapsto (f(v), w - f(v))$, where $v = (gf)^{-1}(g(w))$. Note that $w - f(v) \in \operatorname{Ker} g$.

**3.9** Let $A$ be a ring and $V$ be an indecomposable $A$-module which is artinian as well as noetherian. Then $\mathrm{End}_A V$ is a local ring whose Jacobson-radical is a nil-ideal. (**Hint :** Let $f \in \mathrm{End}_A V$. There exists a $m \in \mathbb{N}$ with $\mathrm{Ker}\, f^n = \mathrm{Ker}\, f^m$ and $\mathrm{Img} f^n = \mathrm{Img} f^m$ for all $n \geq m$. Then $V = \mathrm{Ker}\, f^m \oplus \mathrm{Img} f^m$ and it follows that $f$ is nilpotent or bijective.)

R **3.10** (Theorem of Krull–Schmidt) Let $A$ be a ring and $V$ be an $A$-module which is artinian as well as noetherian. Then $V$ is a direct sum of indecomposable submodules $V_1, \ldots, V_n$. If $V = W_1 \oplus \cdots \oplus W_m$ is another direct sum decomposition of $V$ into indecomposable submodules, then $m = n$, and there exists a permutation $\sigma \in \mathfrak{S}_n$ with $V_i \cong W_{\sigma(i)}$. (**Remark :** The uniqueness assertion in the *Structure Theorem for Finitely Generated Abelian Groups* follows immediately from the Theorem of Krull–Schmidt.)