MA 312 Commutative Algebra / Jan–April 2020

(BS, Int PhD, and PhD Programmes)

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Lectures : Tuesday and Thursday ; 15:30–17:00 **Venue:** MA LH-5 / LH-1

6. Finite Algebras over a Field — Hilbert's Nullstellensatz

Submit a solution of ANY ONE of the *Exercise ONLY. **Due Date : Thursday, 12-03-2020**

Complete Correct Solutions of the **Exercise carry SEVERAL BONUS POINTS !
Recommended to solve the violet colored ^{R}Exercises. Good Seminar Topics!

6.1 Show that each of the following set is *not* an algebraic set

(1) $\{(x,y)\in\mathbb{A}^2_{\mathbb{R}} \mid y=\sin x\}$. (2) $\{(x,y)\in\mathbb{A}^2_{\mathbb{R}} \mid y=\cos x\}$.
(3) $\{(x,y)\in\mathbb{A}^2_{\mathbb{R}} \mid y=e^x\}$. (4) $\{(z,w)\in\mathbb{A}^2_{\mathbb{C}} \mid |z|^2+|w|^2=1\}$.
(5) $\{(\cos t,\sin t,t)\in\mathbb{A}^3_{\mathbb{R}} \mid t\in\mathbb{R}\}$.
(6) $\bigcup_{m\in\mathbb{N}} \mathrm{L}_m$, where L_m is the line $\mathrm{V}(Y-mX)$.

(This shows that arbitrary (in fact, even countable) union of algebraic sets need not be an algebraic set. — **Hint :** Use the Exercise 6.4 (b) below.)

6.2 Show that each of the following set is an algebraic set and find generators for the ideals of algebraic sets in (a), (c) and (d).

(a) Finite subsets of $\mathbb{A}^n_K$, $\in\mathbb{N}^+$. **(b)** $\{(\cos t,\sin t)\in\mathbb{A}^2_{\mathbb{R}}) \mid t\in\mathbb{R}\}$.

(c) (Twisted cubic curve) $\{(t,t^2,t^3)\in\mathbb{A}^3_K \mid t\in K\}$.

(d) $\{(t^m,t^n)\in\mathbb{A}^2_{\mathbb{C}} \mid t\in\mathbb{C}\}$, where m, n are relatively prime positive integers.

6.3 Let K be an arbitrary field and $m,n\in\mathbb{N}^+$.

(a) If we identify $\mathbb{A}^2_K$ with $\mathbb{A}^1_K\times\mathbb{A}^1_K$ in a natural way, show that the Zariski topology on $\mathbb{A}^2_K$ is not the product of the Zariski topologies on the two copies of $\mathbb{A}^1_K$. Compare these two topologies.

(b) The Zariski topology on $\mathbb{A}^n_K$ is Hausdroff if and only if K is finite.

(c) The Zariski topology on $\mathbb{A}^n_{\mathbb{R}}$ (resp. $\mathbb{A}^n_{\mathbb{C}}$) is weaker than the usual topology on $\mathbb{A}^n_{\mathbb{R}}$ (resp. $\mathbb{A}^n_{\mathbb{C}}$).

(d) If $m\leq n$ and we identify $\mathbb{A}^m_K$ as a subset of $\mathbb{A}^n_K$ via the natural inclusion $\varphi:\mathbb{A}^m_K\to\mathbb{A}^n_K$ given by $\varphi(a_1,\ldots,a_m)\mapsto(a_1,\ldots,a_m,0,\ldots,0)$. Then the Zariski topology on $\mathbb{A}^m_K$ is the relative topology from the Zariski topology on $\mathbb{A}^n_K$. Moreover, if W is an algebraic set in $\mathbb{A}^m_K$ then $\varphi(W)$ is an algebraic set in $\mathbb{A}^n_K$. What is the relation between the ideals $I_K(W)$ and $I_K(\varphi(W))$?

(e) Give an example to show that the image of an algebraic set under the natural projection map $\mathbb{A}^2_K\to\mathbb{A}^1_K$ need not be an algebraic set.

6.4 Let L be a line, $H=\mathrm{V}(f)$ be a hypersurface and V be an algebraic set in $\mathbb{A}^n_K$. Then :

(a) Either $\mathrm{L}\subseteq H$ or $\mathrm{L}\cap H$ is a finite set of at most $d=\deg f$ points.

(b) Either $\mathrm{L} \subseteq V$ or $\mathrm{L} \cap V$ is a finite set of points. (How many!)

(c) Let $\mathcal{C} = \mathrm{V}(f)$ and $\mathcal{D} = \mathrm{V}(g)$ be two plane curves in $\mathbb{A}^2_K$. If f and g are relatively prime in $K[X_1, X_2]$ then show that $\mathcal{C} \cap \mathcal{D}$ is a finite set of at most $(\deg f) \cdot (\deg g)$. (**Hint:** Reduce to the case $f \in K[X_1]$ and $g \in K[X_2]$ and then use part (a).)

6.5 Let $L\,|\,K$ be a field extension and let V, $W_1, \ldots, W_r \subseteq \mathbb{A}^n_L$ be algebraic K-sets with W_i irreducible and $W_i \not\subseteq V$ for every $i = 1, \ldots, r$. Then there exists a polynomial $f \in K[X_1, \ldots, X_n]$ such that f vanishes on V but not on any W_i, $i = 1, \ldots, r$. (**Hint:** Use the Exercise 1.9 on the Prime Avoidance.)

6.6 Let $\mathbb{F}_q$ be a *finite* field with q elements and let $f, g \in \mathbb{F}_q[X_1, \ldots, X_n]$.

(a) If $\deg_{X_i}(f) \leq q-1$ for every $i = 1, \ldots, n$ and $f(a) = 0$ for every $a \in \mathbb{F}^n_q$ then $f = 0$.

(b) There exists a unique polynomial $R(f) \in \mathbb{F}_q[X_1, \ldots, X_n]$ such that:

(b.1) $\deg_{X_i}(R(f)) \leq q-1$ for all $i = 1, \ldots n$.

(b.2) $\deg(R(f)) \leq \deg(f)$.

(b.3) $R(f+g) = R(f) + R(g)$.

(b.4) The polynomial function $f - R(f) : \mathbb{F}^n_q \to \mathbb{F}_q$ is the zero function, i.e. $f(a) = R(f)(a)$ for every $a \in \mathbb{F}^n_q$.

(c) (Chevalley's Theorem) If $0 \in \mathrm{V}_{\mathbb{F}_q}(f)$ and if $n > \deg(f)$, then $\mathrm{V}_{\mathbb{F}_q}(f)$ has a non-trivial $\mathbb{F}_q$-rational point $a \in \mathbb{F}^n_q$, $a \neq 0$. (**Proof:** Suppose on the contrary that $\mathrm{V}_{\mathbb{F}_q}(f) = \{0\}$.— Use the part (b) to the polynomial $F = 1 - f^{q-1}$ and use (b.2), (b.4) and (a) to conclude that $R(F) = \prod_{i=1}^n (1 - X_i^{q-1})$. Now, use (b.2) to get:

$$(q-1) \cdot \deg(f) = \deg(F) \geq \deg(R(F)) = \deg(\textstyle\prod_{i=1}^n (1 - X_i^{q-1})) = (q-1) \cdot n$$

and so $\deg(f) \geq n$. a contradiction. •)

(d) If f is homogeneous of degree 2 and if $n \geq 3$, then $\mathrm{V}_K(f)$ has a non-trivial K-rational point. (**Hint:** Use Chevalley's Theorem in the part (c).)

6.7 Let $L\,|\,K$ be a field extension. A K-algebraic set $V \subseteq L^n$ is called a K-cone (with vertex at the origin) if $V = \mathrm{V}_L(F_1, \ldots, F_r)$ for some homogeneous polynomials $F_1, \ldots, F_r \in K[X_1, \ldots, X_n]$. For an algebraic set $V \subseteq K^n$, show that V is a cone if and only if for each $a \in V$, $a \neq 0$, the line $\mathrm{L}(a, 0)$ joining a and 0 is contained in V.

*__6.8__ Let K be an arbitrary field.

(a) If K is infinite then $\mathrm{I}_K(\mathbb{A}^n_K) = 0$. In particular, if K is infinite, then $\mathbb{A}^n_K$ is irreducible.

(b) If K is finite then find a set of generators for the ideal $\mathrm{I}_K(\mathbb{A}^n_K)$. Deduce that if K is finite, then $\mathbb{A}^n_K$ is not irreducible. (**Hint:** Use Exercise 6.6.)

6.9 Let $L\,|\,K$ be a field extension and $V \subseteq L^n$ be an L-algebraic set. Then the set $V_K := V \cap K^n$ of all K-rational points of V is an K-algebraic set in K^n.

6.10 Let K be an *infinite* and let $f_1, \ldots, f_m \in K[X_1, \ldots, X_n]$. If $\mathrm{V}_K(f_1) \cup \cdots \mathrm{V}_K(f_m) = K^n$, then $\mathrm{V}_K(f_1) = K^n$ (or equivalently, $f_i = 0$) for some $i \in \{1, \ldots, m\}$, (**Remark:** One can use this Exercise to prove the Primitive Element Theorem (due to Abel) which states that: *Suppose that $L\,|\,K$ is an algebraic field extension of an infinite field K with $L = K(x_1, x_2, \ldots, xn)$ where*

$x_2,\ldots,x_n$ are separable over K. Then there exists an element $y\in L$ of the form $y=a_1x_1+\cdots+a_nx_n$ with $a_a,\ldots,a_n\in K$ such that $L=K(y)$.)

6.11 Let K be a field.

(a) Let A be a K-algebra, $a_1,\ldots,a_n\in K$ be distinct elements and let $x\in A$ be such that $x-a_1,\ldots,x-a_n$ are units in A. Then $1,x,\ldots,x^{n-1}$ are linearly independent over K if and only if the elements $(x-a_1)^{-1},\ldots,(x-a_n)^{-1}$ are linearly independent over K. (**Hint:** Put $y_i=(x-a_i)^{-1}$ and $y:=\prod_{i=1}^{n}(x-a_i)$. Then $y\in A^\times$ and if $y_1,\ldots,y_n$ are linearly independent over K, then $yy_1,\ldots,yy_n$ linearly independent over K in $K+Kx+\cdots Kx^{n-1}$. Conversely, if $1,x,\ldots,x^{n-1}$ are linearly independent over K and if $b_1y_1+\cdots+b_ny_n=0$ with $b_i\in K$, then multiply by y and compute the co-efficient of x^{n-1} to get $b_1+\cdots+b_n=0$. Therefore $0=\sum_{i=1}^{n}b_i(y_i-y_n)=\sum_{i=1}^{n-1}b_i(a_i-a_n)y_iy_n$ and so $y_1,\ldots,y_n$ are linearly independent over K by induction on n.)

(b) The Hilbert's Nullstellensatz (HNS3) can be easily proved for uncountable fields (for example, for $\mathbb{R}$ and $\mathbb{C}$) as follows :

Let K be a countable field and $L=K[x_1,\ldots,x_n]$ be a field which is finite type over K. If $x\in L$ is not algebraic over K, then the elements $(x-a)^{-1}$, $a\in K$, are K-linearly independent over K. (**Hint:** Use part (a).) On the other hand $\mathrm{Dim}_K L$ is countable. (**Remark:** Analogously one proves : Let K be a uncountable field and L be a field. If L is generated as an K-algebra by x_i, $i\in I$, with $\mathrm{Card}\, I<\mathrm{Card}\, K$. Then every $x\in L$ is algebraic over K.)

6.12 Let $L\,|\,K$ be a field extension with L infinite. For $f_1,\ldots,f_n\in K[T_1,\ldots,T_m]$, put

$$V_0:=\{(f_1(t_1,\ldots,t_m),\ldots,f_n(t_1,\ldots,t_m))\in\mathbb{A}_L^n\mid(t_1,\ldots,t_m)\in\mathbb{A}_L^m\}\,.$$

(a) Show by an example that V_0 need not be an K-algebraic set.

(b) Show that the closure V in $\mathbb{A}_L^n$ (in the Zariski topology) of the set V_0 is an irreducible K-algebraic set. (**Hint:** In fact $V=\mathrm{V}(\mathrm{Ker}\,\varepsilon_f)$, where $\varepsilon_f: K[X_1,\ldots,X_n]\to K[T_1,\ldots,T_m]$, $X_i\mapsto f_i$ for every $i=1,\ldots,n$, is the substitution K-algebra homomorphism. — In this situation one says that V is given by a polynomial parametrization with parameters $T_1,\ldots,T_m$. If $m=1$ and $f_i=T^{d_i}$, $i=1,\ldots,n$, for some positive integers $d_1,\ldots,d_n\in\mathbb{N}^+$ then we say that V is a monomial curve given by the sequence $d_1,\ldots,d_n$ of positive integers.)

(c) Assume that $K=L$ is algebraically closed and $K[T_1,\ldots,T_m]$ is integral over $K[f_1,\ldots,f_n]$, then show that V_0 is closed, that is, $V_0=V$.

6.13 (HNS 4) Let A be a K-algebra of finite type over a field K and let $L\,|\,K$ be a field extension with L algebraically closed. Then $\mathrm{Hom}_{K\text{-alg}}(A,L\neq\emptyset$, i. e. there exits a K-algebra homomorphism $A\to L$. Moreover, prove HNS 1 if and only if HNS 4. (**Remark:** In the case when A is an integral domain, one can even demand more, namely : For given non-zero elements $f_1,\ldots,f_r\in A$, there exists a K-algebra homomorphism $\varphi: A\to L$ such that $\varphi(f_i)\neq 0$ for every $i=1,\ldots,r$. For this proof consider the finite type K-algebra $A[1/f_1,\ldots,f_r]$.)

6.14 Let K be a field and let $\overline{K}$ be a fixed algebraic closure of K.

(a) Every maximal ideal $\mathfrak{m}\in\mathrm{Spm}\;K[X_1,\ldots,X_n]$, there exists $a=(a_1,\ldots,a_n)\in\overline{K}^n$ with $\mathfrak{m}=\mathfrak{m}_a:=\{f\in K[X_1,\ldots,X_n]\mid f(a)=0\}$.

(b) (HNS 5) Let K be an algebraically closed field. Then the map

$$K^n\longrightarrow\mathrm{Spm}\;K[X_1,\ldots,X_n],\; a\longmapsto\mathfrak{m}_a=\langle X_1-a_1,\ldots,X_n-a_n\rangle$$

is bijective. Moreover, for any ideal $\mathfrak{a}\in\mathfrak{I}(K[X_1,\ldots,X_n])$, $a\in\mathrm{V}_K(\mathfrak{a})$ if and only if $\mathfrak{a}\subseteq\mathfrak{m}_a$.

6.15 Let $E\,|\,K$ be an arbitrary field extension and $\mathfrak{a}\subsetneq K[X_1,\ldots,X_n]$ be a non-unit ideal. Then the extended ideal $\mathfrak{a}E[X_1,\ldots,X_n]\subsetneq E[X_1,\ldots,X_n]$ is also a non-unit ideal. (**Hint:** Apply

HNS1 to the field extension $\overline{E}\,|K$, where $\overline{E}$ denote an algebraic closure of E. See also Exercise 5.7 (b).) Moreover, use this result to prove the equivalence of HNS 5 (see Exercise 6.14 (b)) and HNS 1.

6.16 Let K be any field. Then every maximal ideal $\mathfrak{m} \in \operatorname{Spm} K[X_1,\ldots,X_n]$ is generated by n irreducible polynomials $f_1,\ldots,f_n \in K[X_1,\ldots,X_n]$. (**Hint :** Let $\overline{K}$ be a fixed algebraic closure of K. Then by the above Exercise 6.14 (a), $\mathfrak{m} = \mathfrak{m}_a := \{f \in K[X_1,\ldots,X_n] \mid f(a) = 0\}$ with $a = (a_1,\ldots,a_n) \in \overline{K}^n$. Now, choose $f_n \in K[X_1,\ldots,X_{n-1}][X_n]$ with $f_n(a_1,\ldots,a_{n-1},X_n] = \mu_{a_n\,,K[a_1,\ldots,a_{n-1}]}$ (the minimal polynomial of a_n over the field $K[a_1,\ldots,a_{n-1}]$). Let $f \in \mathfrak{m}$. Division with remainder over $K[X_1,\ldots,X_{n-1}]$ yields $f = q\cdot f_n + r$ with $r \in K[X_1,\ldots,X_{n-1}][X_n]$ whose coefficients belong to the kernel $\mathfrak{m}_{n-1}$ of the substitution homomorphism $\varepsilon : K[X_1,\ldots,X_{n-1}] \to K[a_1,\ldots,a_{n-1}]$. By induction we may assume that $\mathfrak{m}_{n-1}$ is generated by $f_1,\ldots,f_{n-1}$. Then $f \in \langle f_1,\ldots,f_{n-1},f_n\rangle$.
— **Remark :** It is even true that every *reduced* ideal $\mathfrak{a}$ in $K[X_1,\ldots,X_n]$ is generated by n polynomials $f_1,\ldots,f_n \in K[X_1,\ldots,X_n]$. Kronecker proved that an K-algebraic set $V \subseteq \overline{K}^n$ can be defined (set-theoretically) by $n+1$ polynomials, i. e. $V = \mathrm{V}(f_1,\ldots,f_{n+1})$ — one can actually replace $n+1$ by n as proved by U. Storch. It is interesting — and difficult — question to determine under what conditions an K-algebraic set $V \subseteq \overline{K}^n$ is defined set-theoretically by $n - \dim V$ equations. Even in the case of *space curves* in $\mathbb{C}^3$, i. e. $\dim V = 1$ and $n = 3$, this is not known.)

6.17 Let K be a field and let $\mathfrak{m},\mathfrak{m}_1,\ldots,\mathfrak{m}_r \in \operatorname{Spm} K[X_1,\ldots,X_n]$ be a maximal ideals.

(a) The K-algebraic set $\mathrm{V}_K(\mathfrak{m}) \subseteq \mathbb{A}^n_K$ contains at most one point. Moreover, $\mathrm{V}_K(\mathfrak{m}) \neq \emptyset$ if and only if if $\mathfrak{m} \in K\text{-Spec } K[X_1,\ldots,X_n]$.

(b) There exists $f_i \in K[X_1,\ldots,X_i]$, $1 \le i \le n$, such that $\mathfrak{m}$ is generated by $f_1,\ldots,f_n$. (**Hint :** Induction on n. For each $i = 1,\ldots,n$, $\mathfrak{m} \cap K[X_1,\ldots,X_i]$ is a maximal ideal in $K[X_1,\ldots,X_i]$ by HNS 3. Use Induction on n. See also Exercise 6.16.)

(c) More generally, there exists $f_i \in K[X_1,\ldots,X_i]$, $1 \le i \le n$, such that the ideal $\mathfrak{a} := \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_r$ is generated by $f_1,\ldots,f_n$. In particular, the ideal $\mathrm{I}_K(\{P_1,\ldots,P_r\})$ of a finite subset $\{P_1,\ldots,P_r\} \subseteq \mathbb{A}^n_K$ is generated by n polynomials. By Chinese Remainder Theorem $K[X_1,\ldots,X_n]/\mathfrak{a} \cong \prod_{i=1}^r K[X_1,\ldots,X_n]/\mathfrak{m}_i$ and $K[X_1,\ldots,X_n]/\ \mathrm{I}_K(\{P_1,\ldots,P_r) \cong K^r$ as K-algebras.

**6.18 Let $\mathfrak{a}$ be an ideal in a polynomial ring $K[X_1,\ldots,X_n]$ over a field K and let $A := K[x_1,\ldots,x_n] = K[X_1,\ldots,X_n]/\mathfrak{a}$.

(a) Show that the ideal $\mathrm{I}_K(\mathrm{V}_K(\mathfrak{a}))/\mathfrak{a}$ in A is the intersection $K\text{-}\mathfrak{r}_A := \bigcap_{\xi \in K\text{-Spec}\,A} \mathfrak{m}_\xi$ of the maximal ideals $\mathfrak{m}_\xi$ corresponding to the homomorphisms (points) $\xi \in K\text{-Spec } A$ and that the equality $\mathrm{I}_K(\mathrm{V}_K(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ is equivalent to the condition that the nilradical of A and the K-radical of A coincide, i. e. $\mathfrak{n}_A = K\text{-}\mathfrak{r}_A$. (**Hint :** Use the identifications

$$\begin{tikzcd} K^n \arrow[r, leftrightarrow] & \operatorname{Hom}_{K\text{-alg}}(K[X_1,\ldots,X_n]\,,K) \arrow[r, leftrightarrow] & K\text{-Spec } K[X_1,\ldots,X_n]\,, \\ a \arrow[r, leftrightarrow] & \xi_a : X_i \mapsto a_i\,, i = 1,\ldots,n \arrow[r, leftrightarrow] & \mathfrak{m}_a = \operatorname{Ker}\xi_a\,. \end{tikzcd}$$

to note that $\mathrm{V}_K(\mathfrak{a}) = \operatorname{Hom}_{K\text{-alg}}(A,K) = K\text{-Spec } A$. — **Remark :** The ideal $K\text{-}\mathfrak{r}_A$ is an invariant of the K-algebra A, called the K-radical of A. Therefore the equality $\mathrm{I}_K(\mathrm{V}_K(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ implies the equality $\mathrm{I}_K(\mathrm{V}_K(\mathfrak{b})) = \sqrt{\mathfrak{b}}$ for any ideal $\mathfrak{b}$ in a polynomial algebra $K[Y_1,\ldots,Y_m]$ with $A \cong K[Y_1,\ldots,Y_m]/\mathfrak{b}$.)

(b) Suppose that K is infinite. Let $f \in K[X,Y]$ be a prime polynomial which is monic Y with coefficients in $K[X]$. Show that $\mathrm{I}_K(\mathrm{V}_K(f)) = (f)$ if and only if $\mathrm{V}_K(f)$ has infinitely many points. (**Hint :** Use the finite free ring extension $K[X] \hookrightarrow K[X,Y]/(f)$ of integral domains to prove that $K\text{-}\mathfrak{r}_A \neq 0$ if and only if $\mathrm{V}_K(f)$ is finite and use the part (a). — **Remark :** Of course, if K

is algebraically closed and if f is a prime polynomial in $K[X,Y]$, then $\mathrm{V}_K(f)$ is infinite and hence $\mathrm{I}_K(\mathrm{V}_K(f)) = (f)$. This is a very special case of HNS2.)

6.19 Let K be a field. If the unit group $K^\times$ of K is finitely generated, then K is finite. (**Remarks:** One can generalize this result to commutative rings which has only finitely many maximal ideals. — Such rings are called semi-local. Let $L|K$ be a finite field extension of an infinite field K. Then the quotient group $L^\times/K^\times$ of the multiplicative groups of L and K is not even finitely generated. This is a much deeper result. However, it is much easier to prove that the quotient group $L^\times/K^\times$ is finite. See also Exercise 4.15.)

6.20 Let K be a field. A commutative K-algebra of finite type in aritinian if and only if it is finite over K. (**Hint:** Use HNS 3.)

6.21 **(a)** A finite commutative reduced $\mathbb{C}$-algebra $\neq 0$ is isomorphic to a product algebra $\mathbb{C}^n$, $n \in \mathbb{N}$, where n is determined uniquely by the isomorphism type of the algebra. Every such a $\mathbb{C}$-algebra is cyclic.

(b) A finite commutative $\mathbb{R}$-algebra $\neq 0$ is isomorphic to a product algebra $\mathbb{R}^m \times \mathbb{C}^n$, $m, n \in \mathbb{N}$, where the natural numbers m, n are determined uniquely by the isomorphism type of the algebra. Every such $\mathbb{R}$-algebra is cyclic.

6.22 Let K be a field. If K is finite type over $\mathbb{Z}$, then K is finite. (**Hint:** If $\operatorname{Char} K = 0$, then show that $\mathbb{Q}$ is finite type over $\mathbb{Z}$-algebra.)

6.23 Let $\mathbb{Z}^n := \{(a_1, \ldots, a_n) \mid a_i \in \mathbb{Z} \text{ for every } i = 1, \ldots, n\}$ be the set of lattice points. If V is an algebraic set in $\mathbb{C}^n$ with $\mathbb{Z}^n \subseteq V$, then $V = \mathbb{C}^n$.

6.24 The aim of this Exercise is to prove the following (generalization of solutions of homogeneous system of linear equations, see also Chevalley's Theorem in Exercise 6.6 (c)) : *Let K be an algebraically closed field and $f_1, \ldots, f_m \in K[X_1, \ldots, X_n]$ be a system of homogeneous polynomials in more indeterminates than the number of equations,* i. e. $n \geq m$. *Then* $\mathrm{V}_K(f_1, \ldots, f_m) \neq \{0\}$. More precisely, for polynomials $f_1, \ldots, f_m \in K[X_1, \ldots, X_n]$ with $\mathrm{V}_K(f_1, \ldots, f_m) = \{0\}$, prove that (the following steps are due to H.-J. Nastold):

(a) There exists a natural number $q \in \mathbb{N}$ such that $X_i^q \in \langle X_1, \ldots, X_n \rangle$ for all $i = 1, \ldots, n$.

(b) Let $q \in \mathbb{N}$ be as in the part (a). If $f_1, \ldots, f_m$ are homogeneous, then for each $i = 1, \ldots, n$, there exist homogeneous polynomials $h_1, \ldots, h_m \in K[X_1, \ldots, X_n]$ of degrees $< q$ such that $X_i^q = h_1 f_1 + \cdots + h_m f_m$.

(c) If $f_1, \ldots, f_m$ are homogeneous, then the ring extension $K[f_1, \ldots, f_m] \subseteq K[X_1, \ldots, X_n]$ id finite, i. e. $K[X_1, \ldots, X_n]$ is a finite $K[f_1, \ldots, f_m]$-algebra. lsmall(**Hint:** By the part (b), every monomial of degree $\geq nq$ can be generated by monomials of lower degree.)

(d) If $f_1, \ldots, f_m$ are homogeneous, then $m \geq n$.

6.25 Let K be a field which is *not* algebraically closed.

(a) For every $m \in \mathbb{N}_+$, there exists a non-constant polynomial $f_m \in K[X_1, \ldots, X_m]$ whose zero-set in K^m is singleton $\{0 = (0, \ldots, 0)\}$, i.e. $\mathrm{V}_K(f) = \{(0, \ldots, 0)\}$.

(b) Every K-algebraic set $V \subseteq K^n$, $n \geq 1$, is a hypersurface in K^n, i. e. it is the zero-set of a single polynomial : $V = \mathrm{V}_K(f)$ with $f \in K[X_1, \ldots, X_n]$. (**Hint:** Use the part (a).)

****6.26** (Generalisation of HNS1) Let K be an arbitrary field, S be the set of all polynomials in $K[X_1, \ldots, X_n]$ which have no zeros in K^n, i. e.

$$S := \{f \in K[X_1,\ldots,X_n] \mid \mathrm{V}_K(f) = \emptyset\}$$

and let $\mathfrak{a}$ be an ideal in $K[X_1,\ldots,X_n]$. If $S \cap \mathfrak{a} = \emptyset$, then $\mathrm{V}_K(\mathfrak{a}) \neq \emptyset$. (**Hint:** Use the Exercise 6.25 (b).)

6.27 Let K be a field. Two elements $x, y \in A$ in the K-algebra A are said to be K-conjugates or conjugate over K if they are algebraic over K and if they have the same minimal polynomial over K, i. e. $\mu_{x,K} = \mu_{y,K}$.

(a) Let $L\,|\,K$ be a *normal* field extension. Then $x,\ y \in L$ are conjugate over K if and only if there exists a K-algebra automorphism $\psi : L \to L$ such that $\psi(x) = y$.

(b) Let $L\,|\,K$ be a *normal* field extension and let L_1 be an intermediary field such that every polynomial in $K[X]$ which has a zero in L has a zero in L_1. Then $L = L_1$. (**Hint:** We may assume that $L\,|\,K$ is finite. If K is finite, then the assertion from that fact that L has a primitive element, i. e. $L = K(x)$ for some $x \in L$. Now, if K is infinite and if $\varphi_1, \ldots, \varphi_r \in \mathrm{Aut}_{K\text{-alg}} L$ are all K-automorphisms of L, then $L = \bigcup_{i=1}^{r} \varphi_i(L_1)$ by the part (a) and hence $L = L_1$.)

6.28 Let $L\,|\,K$ be a *normal* field extension. Two points $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n) \in L^n$, $n \in \mathbb{N}^+$ are K-conjugates if and only if there exists a K-automorphism $\sigma : L \to L$ of L such that $\sigma(b_i) = a_i$ for every $i = 1, \ldots n$.

(a) Let $V \subseteq L^n$ be an K-algebraic set. If $a \in V$, then V contains all K-conjugates of a.

(b) Let $V \subseteq L^n$ be a finite set of points with the property that : if $a \in V$, then V contains all K-conjugates of a. Then V is a K-algebraic set. (**Hint:** If $a \in L^n$, then there exist an ideal $\mathfrak{a} \subseteq K[X_1,\ldots,X_n]$ and a K-algebra isomorphism $K[a_1,\ldots,a_n] \xleftarrow{\sim} K[X_1,\ldots,X_n]/\mathfrak{a}$.)

$^{\mathrm{R}}$**6.29** The $\mathbb{R}$-algebra $\mathrm{C} := \mathbb{R}[X,Y]/\langle X^2+Y^2-1\rangle$ is not UFD, The maximal ideals corresponding to the real points need two generators. The maximal ideals corresponding to the complex points are principal ideals.

(**Hint:** The real circle algebra $\mathrm{C} := \mathbb{R}[X,Y]/(X^2+Y^2-1)$ with the $\mathbb{R}$-spectrum $\mathbb{R}\text{-}\mathrm{Spec}\,\mathrm{C} = \mathrm{S}^1 = \{(a,b) \in \mathbb{R}^2 \mid a^2+b^2 = 1\}$ is not factorial. The maximal ideals $\mathfrak{m}_{(a,b)} = (x-a, y-b)$ corresponding to the points $(a,b) \in S^1$, need two generators. Suppose that the maximal ideal $\mathfrak{m}_{(a,b)} \subseteq \mathrm{C}$ is generated by $f \in \mathrm{C}$. Then f induces a real-valued analytic function on S^1 which has a simple zero at (a,b) and no other zeros. Hence, f changes the sign at (a,b) and has no zero on $\mathrm{S}^1 \smallsetminus \{(a,b)\} \cong \mathbb{R}^1$. This contradicts the intermediate value theorem.

For an algebraic proof, consider the quadratic $\mathbb{R}[x]$-algebra $\mathrm{C} = \mathbb{R}[x][y]$ which is free with basis $1, y$ over the polynomial algebra $\mathbb{R}[x] \subseteq \mathrm{C}$ with defining equation $y^2 = 1 - x^2$. Then the norm (see the Footnote No 4) of an element $g = g_0 + g_1 y$, $g_0, g_1 \in \mathbb{R}[x]$, is $\mathrm{N}(g) = \mathrm{N}^{\mathrm{C}}_{\mathbb{R}[x]}(g) = g_0^2 + g_1^2(x^2-1)$. In particular, $\mathrm{N}(y) = x^2 - 1$ and, either $\mathrm{N}(g) \in \mathbb{R}^\times$ or $\deg(\mathrm{N}(g)) \geq 2$. Now, if $f \in \mathrm{C}$ generates $\mathfrak{m}_{(a,b)}$, then $\mathrm{N}(f)$ has to be a linear polynomial, since $\mathrm{Dim}_{\mathbb{R}}(\mathbb{R}[x]/\mathbb{R}[x]\,\mathrm{N}(f)) = \mathrm{Dim}_{\mathbb{R}}(\mathrm{C}/\mathrm{C}f) = 1$ which is impossible (the leading term of $\mathrm{N}(f)$ is of even degree). However, the square $\mathfrak{m}^2_{(a,b)} = \langle (x-a)^2, (x-a)(y-b), (y-b)^2\rangle = \langle ax+by-1\rangle$ is a principal ideal.)

$^{\mathrm{R}}$**6.30** The $\mathbb{R}$-algebra $\mathrm{M} := \mathbb{R}[X,Y]/\langle X^2+Y^2+1\rangle$ is a PID, but not an Euclidean domain[1].

[1] **Euclidean functions and Euclidean domains** Let A be an integral domain. A *Euclidean function* on A is a map $\delta : A \smallsetminus \{0\} \to \mathbb{N}$ which satisfies the following property : for every two elements $a, b \in A$ with $b \neq 0$ there exist elements q and r in A such that $a = qb + r$ and either $r = 0$ or $\delta(r) < \delta(b)$. If there is a Euclidean function δ on A, then A is called a *Euclidean domain* (with respect to δ). For example, the usual absolute value function $|\cdot| : \mathbb{Z} \smallsetminus \{0\} \to \mathbb{N}$, $a \mapsto |a|$ is a Euclidean function on the the ring of integers $\mathbb{Z}$; For a field K, the degree function

(**Remark :** The maximal Spm M contains only complex points and can be identified with the punctured real projective plane, i. e. the Möbius strip.)

(**Hint :** Use the following simple key observation that the K-Spectrum of a K-algebra of finite type A over a field K which is an Euclidean domain with a small unit group is non-empty. More precisely :

Proposition *Let A be an affine domain*[2] *over a field K. If A is an Euclidean domain, then there exists a maximal ideal $\mathfrak{m} \in \operatorname{Spm} A$ such that the natural group homomorphism $\pi^\times : A^\times \to (A/\mathfrak{m})^\times$ (which is the restriction of the canonical surjective map $\pi : A \to A/\mathfrak{m}$) is surjective. In particular, if A is an Euclidean domain with $A^\times = K^\times$, then K-Spec $A \neq \emptyset$.*

Proof : Suppose that A is an Euclidean domain and that $\delta : A \setminus \{0\} \to \mathbb{N}$ is a minimal Euclidean function on A. Then choose an element $f \in A$ such that $\delta(f) := \operatorname{Min}\{\delta(a) \mid 0 \neq a \in \left(A \smallsetminus A^\times\right)\}$. Such an element f exists, since the ordered set $(\mathbb{N}, \leq)$ where $\leq$ is the usual order on $\mathbb{N}$, is well ordered. We claim that f is irreducible. For, if $f = gh$ with $g, h \in A$, then $\delta(f) = \delta(gh) \geq \delta(g)$. In the case $\delta(f) > \delta(g)$, $g \in A^\times$ by the minimality of $\delta(f)$. In the case $\delta(gh) = \delta(f) = \delta(g)$, $h \in A^\times$. Therefore, $\mathfrak{m} = Af$ is a non-zero prime ideal and hence a maximal ideal in A. To prove that $\pi^\times : A^\times \to (A/\mathfrak{m})^\times$ is surjective, let $z \in (A/\mathfrak{m})^\times$. Then $z = \pi(g)$ for some $g \in A$ and $g \notin \mathfrak{m}$. Use the Euclidean function δ to write $g = fq + r$ with $q, r \in A$ and either $r = 0$ or $\delta(r) < \delta(f)$. Since $z \neq 0$, i. e. $g \notin \mathfrak{m}$, we must have $r \neq 0$ and hence $\delta(r) < \delta(f)$. But, then by the minimality of $\delta(f)$, $r \in A^\times$ and $z = \pi(g) = \pi(r) = \pi^\times(r)$. •

We can reformulate the above Proposition in the language of algebraic geometry as :

Corollary *Let $\mathcal{C}$ be an affine algebraic irreducible curve over a field K. If $\mathcal{C}$ has no K-rational points and the unit group of the coordinate ring $K[\mathcal{C}]$ of $\mathcal{C}$ is $K^\times$, then $K[\mathcal{C}]$ is not a Euclidean domain.)*

R **6.31** In most textbooks it is stated that there are examples of principal ideal domains which are not Euclidean domains. However, concrete examples are almost never presented with full details. In this subsection we use HNS 3 to give a family of such examples with full proofs which are accessible even to undergraduate students. The main ingredients are computations of the unit group[3] $A^\times$ of A by using the norm map[4] and the K-Spec A for an affine algebras over a field K.

$f \mapsto \deg f$ is a Euclidean function on the the polynomial ring $K[X]$; the order function $f \mapsto \operatorname{ord} f$, is a Euclidean function on the the formal power series ring $K[\![X]\!]$.

Note that in the definition of a Euclidean function on A, many authors also include the condition that δ respect the multiplication, i. e. $\delta(ab) \geq \delta(a)$ for all $a, b \in A \smallsetminus \{0\}$. However, if A is a Euclidean domain, then there exists a so-called *minimal Euclidean function* δ on A which respects the multiplication and the equality $\delta(ab) = \delta(a)$ for $a, b \in A \smallsetminus \{0\}$ holds if and only if $b \in A^\times$. For a proof we recommend the reader to see the beautiful article by P. Samuel : [Samuel, P. : About Euclidean rings. *J. Algebra* **19** (1971), 282–301.]

In a Euclidean domain, any two elements have a gcd which can be effectively computed by *Euclidean algorithm.* In particular, Euclidean domains are principal ideal domains and hence unique factorization domains.

[2] A K-algebra of finite type over a field K is called an affine algebra over K. An affine algebra over a field K which is an integral domain is called affine domain over K.

[3] **Unit Groups** For a ring A, the group $A^\times$ of the invertible elements in the multiplicative monoid $(A, \cdot)$ of the ring A is called the *unit group* ; its elements are called the *units* in A. The determination of the unit group of a ring is an interesting problem which is not always easy. Some simple examples are : $\mathbb{Z}^\times = \{-1, 1\}$; if $n \geq 2$, then $\mathbb{Z}_n^\times = \{m \in \mathbb{N} \mid 0 \leq m < n \text{ and } \gcd(m,n) = 1\}$; if K is a field then $K^\times = K \smallsetminus \{0\}$; if A is an integral domain, then $(A[X_1, \ldots, X_n])^\times = A^\times$; if K is a field, then $(K[T, T^{-1}])^\times = \{\lambda T^n \mid \lambda \in K^\times \text{and } n \in \mathbb{Z}\} \cong$ the product group $K^\times \times \mathbb{Z}$; $(A[\![X_1, \ldots, X_n]\!])^\times = \{f \in A[\![X_1, \ldots, X_n]\!] \mid f(0) \in A^\times\}$.

[4] **Norm** The notion of the *norm* is very useful for the determination of the unit groups of some domains. Let R be a (commutative) ring and let A be a finite free R-algebra. For $x \in A$, let $\lambda_x : A \to A$ denote the (left) multiplication by x. The *norm* map $\mathrm{N}_R^A : A \to R$, $x \mapsto \operatorname{Det} \lambda_x$, contains important information about the multiplicative structure of A over R. The following properties of the norm map are easy to verify :

The norm map $\mathrm{N}_R^A : A \to R$ is multiplicative, i. e. $\mathrm{N}_R^A(xy) = \mathrm{N}_R^A(x) \cdot \mathrm{N}_R^A(y)$ *for all $x, y \in A$,* $\mathrm{N}_R^A(a) = a^n$ *for every $a \in R$, where $n := \operatorname{Rank}_R(A)$. Further, for an element $x \in A$, $x \in A^\times$ if and only if $\mathrm{N}_R^A(a) \in R^\times$.*

(a) In the following examples we shall illustrate the use of the norm map to compute the unit group.

(1) **Lemma** *Let $\varphi(X)\in\mathbb{R}[X]$ be a non-constant polynomial with positive leading coefficient, $\Phi:=Y^2+\varphi(X)\in\mathbb{R}[X,Y]$ and let $A:=\mathbb{R}[X,Y]/\langle\Phi\rangle$. Then A is an affine domain (over $\mathbb{R}$) of (Krull) dimension 1 and $A^\times=\mathbb{R}^\times$.*

(**Proof** Let $x,y\in A$ denote the images of X,Y in A respectively. Then A is a free $\mathbb{R}[X]$-algebra of rank 2 with R-basis $1,y$, i. e. $A=\mathbb{R}[X]+\mathbb{R}[X]\cdot y$ and $y^2=-\varphi(X)$. Further, let $\mathrm{N}:=\mathrm{N}^A_{\mathbb{R}[X]}:A\to\mathbb{R}[X]$ denote the norm-map of A over $\mathbb{R}[X]$. Then $\mathrm{N}(F+Gy)=\mathrm{Det}\begin{pmatrix}F & -G\varphi\\ G & F\end{pmatrix}=F^2+G^2\varphi$ for every $F,G\in\mathbb{R}[X]$. Therefore $F+Gy\in A^\times$ if and only if $F^2+G^2\varphi\in\mathbb{R}[X]^\times=\mathbb{R}^\times$, equivalently, $F\in\mathbb{R}^\times$ and $G=0$, since the leading coefficient of φ is positive by assumption. This proves that $A^\times=\mathbb{R}^\times$.)

(2) The $\mathbb{R}$-algebras

(i) $\mathrm{P}:=\mathbb{R}[X,Y]/\langle Y^2-X\rangle\cong\mathbb{R}[Y]$,

(ii) $\mathrm{H}:=\mathbb{R}[X,Y]/(X^2-Y^2-1)\cong\mathbb{R}[X,Y]/\langle XY-1\rangle\cong\mathbb{R}[Z,Z^{-1}]$,

(iii) $\mathrm{C}:=\mathbb{R}[X,Y]/\langle X^2+Y^2-1\rangle$,

(iv) $\mathrm{L}_{b,c}:=\mathbb{R}[X,Y]/\langle Y^2+bX^2+c\rangle$ with $b,c\in\mathbb{R},b>0$,

are all affine domains (over $\mathbb{R}$) of dimension one, (i. e. every non-zero prime ideal is maximal) and $\mathrm{H}^\times\cong\mathbb{R}^\times\times\mathbb{Z}$, $\mathrm{P}^\times=\mathrm{K}^\times=\mathrm{L}^\times_{b,c}=\mathbb{R}^\times$. For the $\mathbb{R}$-affine domains H (see (ii) above) and C (see (iii) above), the assumptions in Proposition in Exercise 6.30 are not satisfied, but H is a Euclidean domain and C is not a Euclidean domain, in fact, not even a PID or a UFD, see Exercise 6.29.

(b) Lemma *Let $\varphi(X)\in\mathbb{R}[X]$ be a non-constant polynomial with $\varphi(\alpha)>0$ for every $\alpha\in\mathbb{R}$ and let $\Phi:=Y^2+\varphi(X)\in\mathbb{R}[X,Y]$. Then the affine domain $A:=\mathbb{R}[X,Y]/\langle\Phi\rangle$ is not a Euclidean domain. In particular, $\mathrm{L}_{b,c}=\mathbb{R}[X,Y]/\langle Y^2+bX^2+c\rangle$ with $b,\ c\in\mathbb{R}$, $b>0$, $c>0$ is not a Euclidean domain.*

(**Proof :** Note that $A^\times=\mathbb{R}^\times$ by the part (a) (1) and $\mathbb{R}\text{-Spec}\,A=\{(\alpha,\beta)\in\mathbb{R}^2\mid\Phi(\alpha,\beta)=0\}=\emptyset$ by the assumption on φ. Therefore A can not be a Euclidean domain by Corollary in Exercise 6.30. •)

(c) In the following theorem, we give a criterion for the affine $\mathbb{R}$-domain $\mathrm{L}_{b,c}$ to be a principal ideal domain :

Theorem *Let $b,\ c\in R$, $b>0$ and $c\neq 0$. Then the affine domain $\mathrm{L}_{b,c}:=\mathbb{R}[X,Y]/\langle Y^2+bX^2+c\rangle$ over $\mathbb{R}$ is a principal ideal domain if and only if $c>0$.*

(**Proof :** By replacing X by $\sqrt{|c|/b}X$ and Y by $\sqrt{|c|}Y$, it follows that $\mathrm{L}_{b,c}\cong\begin{cases}\mathrm{L}_{1,1} & \text{if } c>0,\\ \mathrm{L}_{1,-1} & \text{if } c<0,\end{cases}$ as $\mathbb{R}$-algebras and hence we may assume that $b=1$ and $c=\pm 1$. Since $\mathrm{L}_{1,-1}$ is not a principal ideal domain by Exercise 6.29, it is enough to prove that $A:=\mathrm{L}_{1,1}$ is a principal ideal domain. Note that $B:=\mathbb{C}\otimes_{\mathbb{R}}A=\mathbb{C}[X,Y]/\langle X^2+Y^2+1\rangle\xrightarrow{\sim}\mathbb{C}[U,V]/\langle UV-1\rangle\cong\mathbb{C}[T,T^{-1}]$ is a principal ideal domain and that B is a free A-algebra with basis $1,\mathrm{i}$, where $\mathrm{i}\in\mathbb{C}$ with $\mathrm{i}^2+1=0$. Let $x\ ,y\in B$ denote the images of X, Y in B respectively and let $\sigma:B\to B$, $\mathrm{i}\mapsto-\mathrm{i}$, denote the conjugation automorphism of B over A. Then $\sigma^2=\mathrm{id}_B$ and $(x+\mathrm{i}y)\cdot\sigma(x+\mathrm{i}y)=(x+\mathrm{i}y)(x-\mathrm{i}y)=-1$, in particular, $\sigma(x+\mathrm{i}y)=-(x+\mathrm{i}y)^{-1}$. Further, an element $f\in B$ belongs to A if and only if $\sigma(f)=f$. Moreover, $B^\times=\{\lambda(x+iy)^n\mid\lambda\in\mathbb{C}^\times \text{ and } n\in\mathbb{Z}\}$.

Let $\mathfrak{A}$ be any ideal in A. To show that $\mathfrak{A}$ is principal, we may assume that $\mathfrak{A}\neq 0$ and $\mathfrak{A}\neq A$. Since B is a PID, the ideal $\mathfrak{A}B\,(\neq 0)$ generated by $\mathfrak{A}$ in B is principal. We claim that there exists $f\in A$ such that $\mathfrak{A}B=Bf$. First choose $g\in B$, $g\neq 0$ such that $\mathfrak{A}B=Bg$. Since $B\sigma(g)=\sigma(Bg)=\sigma(\mathfrak{A}B)=\sigma(\mathfrak{A})B=$

$\mathfrak{A}B = Bg$ and since B is an integral domain, there exists a unit $u \in B^{\times}$ such that $\sigma(g) = u \cdot g$. Further, since $\sigma^2 = \mathrm{id}_B$ and $g \neq 0$, we have $u \cdot \sigma(u) = 1$. Therefore $u = \lambda(x+\mathrm{i}y)^n$ for some $(\lambda, n) \in \mathbb{C}^{\times} \times \mathbb{Z}$ and $1 = u \cdot \sigma(u) = \lambda(x+\mathrm{i}y)^n \cdot \sigma(\lambda)(-1)^n(x+\mathrm{i}y)^{-n} = (-1)^{n|}\,|\lambda|^2$. This proves that n is even and $|\lambda|^2 = 1$, i. e. $n = 2m$ and $\lambda = e^{\mathrm{i}t}$ with $m \in \mathbb{Z}$ and $t \in \mathbb{R}$.

Now, put $f := \mathrm{i}^m e^{\mathrm{i}t/2}(x+\mathrm{i}y)^m \cdot g$. Then $\mathfrak{A}B = Bg = Bf$. To show that $f \in A$, it is enough to prove that $\sigma(f) = f$. We have

$$\begin{aligned}\sigma(f) = (-\mathrm{i})^m e^{-\mathrm{i}t/2}(x-\mathrm{i}y)^m \cdot \sigma(g) &= (-\mathrm{i})^m e^{-\mathrm{i}t/2} \cdot (x-\mathrm{i}y)^m \cdot u \cdot g\\ &= (-\mathrm{i})^m e^{-\mathrm{i}t/2}(x-\mathrm{i}y)^m \cdot e^{\mathrm{i}t}(x+\mathrm{i}y)^{2m} \cdot g\\ &= (-\mathrm{i})^m e^{it/2}(x-\mathrm{i}y)^m(x+\mathrm{i}y)^m(x+\mathrm{i}y)^m \cdot g\\ &= (-\mathrm{i})^m(-1)^m e^{it/2}(x+\mathrm{i}y)^m \cdot g = \mathrm{i}^m e^{it/2}(x+\mathrm{i}y)^m \cdot g = f\,.\end{aligned}$$

Therefore, since B is a free A-module with basis $1, \mathrm{i}$, it follows that $\mathfrak{A} = \mathfrak{A}B \cap A = Bf \cap A = Af$ is a principal ideal. •)

(d) Finally, we come to a class of affine domains over $\mathbb{R}$ which are principal ideal domains, but not Euclidean domains :

Theorem *Let $b, c \in \mathbb{R}$ with $b > 0$ and $c > 0$ and let $\Phi := Y^2 + bX^2 + c \in \mathbb{R}[X,Y]$. Then the affine $\mathbb{R}$-domain $\mathrm{L}_{b,c} := \mathbb{R}[X,Y]/\langle\Phi\rangle$ is a principal ideal domain and is not an Euclidean domain.*

(**Proof :** By Theorem in part (c) $\mathrm{L}_{b,c}$ is a principal ideal domain and by Lemma in part (b) $\mathrm{L}_{b,c}$ is not a Euclidean domain.)