MA 312 Commutative Algebra / Jan–April 2020

(BS, Int PhD, and PhD Programmes)

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Lectures : Tuesday and Thursday ; 15:30–17:00 **Venue:** MA LH-5 / LH-1

8. The Prime Spectrum[1] — Zariski Topology

• **For a ready reference use the R8 Summary of Results listed below**

R8 Summary of Results

R 8.0 Some topological concepts.

R 8.0.1 Noetherian topological spaces. A topological space X is called *noetherian* if any one of the following equivalent conditions holds :

(i) Every open subset of X is quasi-compact[2].
(ii) The open subsets of X satisfy the ascending chain condition.
(iii) The closed subsets of X satisfy the descending chain condition.
(iv) Every non-empty family of open subsets of X contains a maximal element.
(v) Every non-empty family of closed subsets of X contains a minimal element.

In particular, noetherian topological space is quasi-compact and every subspace $Y \subseteq X$ is also noetherian. Moreover, every non-empty noetherian topological space X is a finite the union of irreducible closed subspaces, see Exercise 8.2. In particular, every noetherian topological space is locally connected.

By the formal Hilbert's nullstellensatz, the prime spectrum $\operatorname{Spec} A$ of a ring A is a noetherain topological space if and only if the radical ideals in A satisfy the ascending chain condition, in particular, if A is a noetherain ring, then the prime ideals satisfy ACC and $X = \operatorname{Spec} A$ is a noetherian topological space. More generally, if $A = \sum_{i=1}^{n} A f_i$ is a finitely generated ideal in a ring A, then $\mathrm{D}(\mathfrak{a}) = \cup_{i=1}^{n} \mathrm{D}(f_i)$ is quasi-compact.

The ring $\mathbb{Q}[X_i \mid i \in \mathbb{N}]/\langle X_i^2 \mid i \in \mathbb{N}\rangle$ is not noetherain, but its prime spectrum $\{\mathfrak{m} := \langle x_i \mid i \in \mathbb{N}\rangle\}$ is singleton and hence noetherian.

R 8.0.2 Irreducible topological spaces. A topological space X is called *irreducible* if it satisfies any one of the following equivalent conditions :

(i) $X \neq \emptyset$ and $U \cap V \neq \emptyset$ for arbitrary non-empty open subsets $U, V \subseteq X$.
(ii) $X \neq \emptyset$ and every non-empty open subset of X is dense in X.
(iii) $X \neq \emptyset$ and every non-empty open subset of X is connected, i .e. it is not a disjoint union of two

[1] **The prime spectrum of a (commutative) ring.** *Modern Algebraic Geometry* is a fascinating branch of Mathematics that combines methods from Commutative Algebra and Geometry. It transcends the limited scope of Commutative Algebra by means of geometrical construction principles. The challenge of new problems has caused extensions and revisions. The concept of *schemes* invented by Grothendieck in the lates 1950s made it possible to introduce methods even into fields that formerly seemed far from Geometry, for example Algebraic Number Theory. This paved the way to spectacular new achievements such as the proof of Fermat's Last Theorem (by Wiles and Taylor) a famous problem that was open for more than 350 years. Commutative algebra is one of the foundation stones of this modern algebraic geometry. It provides the complete local tools for the subject the same way as differential calculus provides the tools for differential geometry.

The first step in the *Language of Schemes* is to explain the construction of so-called *affine schemes* which are schemes of special type, namely $\operatorname{Spec} A$ the *prime spectrum of a (commutative) ring* A. Such schemes serve as the local parts from which more general *global* schemes are obtained via gluing process. The prime spectrum $\operatorname{Spec} A$ of a (commutative) ring A is a ringed space, i. e. a topological space with a sheaf of rings on it. In this Exercise Set we discuss $\operatorname{Spec} A$ as a topological space. Its topology so-called *Zariski topology* was primarily (for closed points) introduced by Ocar Zariski and later generalized to $\operatorname{Spec} A$.

[2] A subset U of a topological space X is called *quasi-compact* if every open cover of U admits a finite subcover.

non-empty open subsets of X..

(iv) $X \neq \emptyset$ and $Y \cup Z \subsetneq X$ for arbitrary proper closed subsets $Y, Z \subsetneq X$.

R 8.1 The K-Spectrum of an algebra over a field K and the K-Zariski Topology. Classical Hilbert's Nullstellensatz is the starting point of classical algebraic geometry which provides a bijective correspondence between affine algebraic sets which are geometric objects and radical ideals in a polynomial ring over an algebraically closed field which are algebraic objects. Below we fix conventions, notations, concepts and some most important results and examples.

R 8.1.1 K-Spectrum of a polynomial algebra and K-Zariski topology. Let $K[X_1,\ldots,X_n]$ be a polynomial algebra over a field K in indeterminates $X_1,\ldots,X_n$.

(a) The map $K^n \to \operatorname{Spm} K[X_1,\ldots,X_n]$, $a=(a_1,\ldots,a_n) \mapsto \mathfrak{m}_a = \langle X_1-a_1,\ldots,X_n-a_n\rangle$, is injective.

(b) The map $K^n \longrightarrow \operatorname{Hom}_{K\text{-alg}}(K[X_1,\ldots,X_n],K)$, $a=(a_1,\ldots,a_n) \longmapsto \xi_a$, is bijective, where $\xi_a : K[X_1,\ldots,X_n] \to K$, $X_i \mapsto a_i$ is the substitution homomorphism. (**Hint :** Use the universal property of the K-algebra $K[X_1,\ldots,X_n]$.)

(c) The subset $K\text{-Spec}\, K[X_1,\ldots,X_n] := \{\mathfrak{m} \in \operatorname{Spm} K[X_1,\ldots,X_n] \mid K[X_1,\ldots,X_n]/\mathfrak{m} = K\}$ of the maximal spectrum $\operatorname{Spm} K[X_1,\ldots,X_n]$ is called the *K-spectrum* of $K[X_1,\ldots,X_n]$.

The map $\operatorname{Hom}_{K\text{-alg}}(K[X_1,\ldots,X_n],K) \longrightarrow K\text{-Spec}\, K[X_1,\ldots,X_n]$, $\xi \longmapsto \operatorname{Ker}\xi$, is bijective.

(**Hint :** Every maximal ideal $\mathfrak{m}$ in $K[X_1,\ldots,X_n]$ with $K[X_1,\ldots,X_n]/\mathfrak{m} = K$ is of the type $\mathfrak{m}_a$ for a unique $a=(a_1,\ldots,a_n) \in K^n$; the component a_i is determined by the congruence $X_i \equiv a_i \mod \mathfrak{m}$. — Therefore $K\text{-Spec}\, K[X_1,\ldots,X_n] := \{\mathfrak{m}_a \in \operatorname{Spm} K[X_1,\ldots,X_n] \mid a \in K^n\}$.)

(d) Use (a) and (b), to establish the identifications :

$$\begin{array}{ccccc} K^n & \longleftrightarrow & \operatorname{Hom}_{K\text{-alg}}(K[X_1,\ldots,X_n],K) & \longleftrightarrow & K\text{-Spec}\, K[X_1,\ldots,X_n]\,, \\ a & \longleftrightarrow & \xi_a & \longleftrightarrow & \mathfrak{m}_a = \operatorname{Ker}\xi_a\,. \end{array}$$

(e) For an ideal $\mathfrak{a}$ in $K[X_1,\ldots,X_n]$, the set of common zeros

$$\mathrm{V}_K(\mathfrak{a}) := \{a \in K^n \mid F(a) = 0 \text{ for all } F \in \mathfrak{a}\} = \textstyle\bigcap_{F\in\mathfrak{a}} \mathrm{V}_K(F)$$

of all polynomials $F \in \mathfrak{a}$ in $K^n = K\text{-Spec}\, K[X_1,\ldots,X_n]$, is called an *affine K-algebraic set* or *affine algebraic K-variety defined by* $\mathfrak{a}$. Note that $\mathrm{V}_K(\mathfrak{a}) = \mathrm{V}_K(\sqrt{\mathfrak{a}})$.

Further, the set $\mathcal{F}_K(K^n) := \{\mathrm{V}_K(\mathfrak{a}) \mid \mathfrak{a} \in \text{r-}\mathfrak{I}(K[X_1,\ldots,X_n])\}$ of all affine K-algebraic sets in K^n form the closed sets of a topology — called the *K-Zariski topology* on $K^n = K\text{-Spec}\, K[X_1,\ldots,X_n]$. The open subsets are the complements $\mathrm{D}_K(\mathfrak{a}) := K^n \smallsetminus \mathrm{V}_K(\mathfrak{a})$, $\mathfrak{a} \in \text{r-}\mathfrak{I}(K[X_1,\ldots,X_n])$ and $\mathrm{D}_K(F) = \{a \in K^n \mid F(a) \neq 0\} = K^n \smallsetminus \mathrm{V}_K(F)$, $F \in K[X_1,\ldots,X_n]$, form a basis for the Zariski topology on K^n.

Therefore we have defined the inclusion reversing map :

$$\mathrm{V}_K : \text{r-}\mathfrak{I}(K[X_1,\ldots,X_n]) \longrightarrow \mathcal{F}_K(K^n), \quad \mathfrak{a} \longmapsto \mathrm{V}_K(\mathfrak{a}).$$

(f) (*Polynomial maps*) For $F \in K[X_1,\ldots,X_n]$, the function $Fi^* : K^n \to K$, $a \mapsto F(a)$, is called the *polynomial function* defined by F. By the identifications in (d) $F(a) = \xi_a(F) \equiv F \bmod \mathfrak{m}_a$ for any $a \in K^n$; $F(a)$ is called the *value of F* at a, or at ξ_a, or at $\mathfrak{m}_a$.

For an infinite field K, the polynomial function φ_F^* defined by F determines the polynomial F. This is the following well-known *identity theorem for polynomials*[3], see Exercise 4.13.

Let $\varphi : K[Y_1,\ldots,Y_m] \to K[X_1,\ldots,X_n]$ be a K-algebra homomorphism and let $F_i := \varphi(Y_i), 1 \le i \le m$. Then the map $\varphi^* : K^n \to K^m$ defined by $\varphi^*(a_1,\ldots,a_n) = (F_1(a),\ldots,F_m(a))$ is called the *polynomial map* associated to φ. Under the identifications in (d), the polynomial map φ^* is obviously described as follows : $\xi_a \mapsto \varphi^*\xi_a = \xi_a \circ \varphi$ or by $\mathfrak{m}_a \mapsto \varphi^*\mathfrak{m}_a = \varphi^{-1}(\mathfrak{m}_a) = \mathfrak{m}_{F(a)}$, $a \in K^n$. For every $G \in K[Y_1,\ldots,Y_m]$, we have $G^* \circ \varphi^* = \varphi(G)^*$.

Examples. Let $\varphi : A \to B$ be a K-algebra homomorphism. If φ is an isomorphism, then φ^* is bijective (with

[3] **Identity Theorem for Polynomials.** *Let K be an infinite field and let $F, G \in K[X_1,\ldots,X_n]$. If $\varphi_F^* = \varphi_G^*$ then $F = G$.*.

$(\varphi^*)^{-1} = (\varphi^{-1})^*$). However, if φ^* is bijective then φ need not be an isomorphism. For example:

(1) Let K be a perfect field of characteristic $p > 0$. Then the Frobenius map $ff_p : K \to K$, $x \mapsto x^p$, is bijective, but the corresponding K-algebra homomorphism $K[X] \to K[X]$, $X \mapsto X^p$, is not an automorphism.

(2) For an odd integer $n > 1$, the map $\mathbb{R} \to \mathbb{R}$, $x \mapsto x^n$, is bijective, but the corresponding $\mathbb{R}$-algebra homomorphism $\mathbb{R}[X] \to \mathbb{R}[X]$, $X \mapsto X^n$, is not bijective. If we replace $\mathbb{R}$ by $\mathbb{C}$ then the map $x \mapsto x^n$ is not bijective.

(**Remark :** This is can be generalized in many ways by using the following important result :

Theorem. *Let K be an algebraically closed field of characteristic zero and let φ be a K-algebra endomorphism of $K[X_1,\ldots,X_n]$. If $\varphi^* : K^n \to K^n$ is bijective, then φ is an automorphism.*

The example (1) above shows that the assumption about the characteristic in this theorem is necessary. The group $\operatorname{Aut}_{Kalg} K[X_1,\ldots,X_n]$ of K-algebra automorphisms of a polynomial algebra $K[X_1,\ldots,X_n]$, $n > 1$ (for $n = 1$, see Exercise 4.11), is not yet well understood. In this connection, let us state a famous Jacobian conjecture which is still open in general.

Jacobian Conjecture. *Let K be a field of characteristic zero, φ be a K-algebra endomorphism of $K[X_1,\ldots,X_n]$ and let $F_i := \varphi(X_i)$, $1 \le i \le n$. Then φ is bijective if (and only if) the Jacobian determinant*

$$\frac{\partial(F_1,\ldots,F_n)}{\partial(X_1,\ldots,X_n)} := \operatorname{Det}\left(\frac{\partial F_i}{\partial X_j}\right)_{1\le i,\,j\le n}$$

is a non-zero constant.)

(g) For a better understanding of the map V_K, we define the inclusion reversing map in the opposite direction :

$$\mathrm{I}_K : \mathcal{F}_K(K^n) \longrightarrow \text{r-}\mathcal{I}(K[X_1,\ldots,X_n]).$$

For this to *every* subset $V \subseteq K^n$, we associate the radical ideal in $K[X_1,\ldots,X_n]$

$$\mathrm{I}_K(V) := \{F \in K[X_1,\ldots,X_n] \mid F(a) = 0 \text{ for all } a \in V\} = \textstyle\bigcap_{a\in V} \mathfrak{m}_a \in \text{r-}\mathcal{I}(K[X_1,\ldots,X_n])$$

is called the K-ideal of V. Further, the reduced affine K-algebra

$$K[V] := K[X_1,\ldots,X_n]/\mathrm{I}_K(V)$$

is called the K-coordinate ring of V which is also called the ring of regular, or polynomial K-valued functions on V because its elements $f = F\,(\mathrm{mod}\ \mathrm{I}_K(V)) \in K[V]$ can be considered as (polynomial) function $f : V \to K$, $a \mapsto F(a)$ (which is independent of the choice of a representative F of the residue-class f). For example, if $x_i = X_i(\mathrm{mod}\ \mathrm{I}_K(V))$, then $x_i : V \to K$, $a \mapsto a_i (= i$-th coordinate of $a)$, $i = 1,\ldots,n$, is called its i-th coordinate function. Therefore the map

$$K[V] \xrightarrow{\ \sim\ } \operatorname{Poly}(V,K), \quad \overline{F} \longmapsto (F : V \to K\,, a \mapsto F(a))$$

is a K-algebra isomorphism.

(h) (Morphism of affine K-algebraic sets) Let K be a field and let $V \subseteq K^m$, $W \subseteq K^n$ be affine K-algebraic sets. A map $f : V \to W$ is called a morphism of affine algebraic K-sets if there exist polynomials $F_1,\ldots,F_n \in K[X_1,\ldots,X_m]$ such that $f(a) = (F_1(a),\ldots,F_n(a)) \in W$ for every $a \in V$. In other words, f is a polynomial map.

Note that the composition of morphisms of affine algebraic K-sets is again a morphism of affine algebraic K-sets. Therefore, the collection of affine algebraic K-sets with morphisms of K-algebraic sets form a category — denoted by $\mathcal{A}\!f\!f\ K\text{-}\mathcal{A}lg\,\mathcal{S}ets$. The set of morphisms in this category is denoted by $\operatorname{Hom}_{\mathcal{A}ff\ K\text{-}\mathcal{A}lg\,\mathcal{S}ets}(V,W)$. In particular, every isomorphism[4] of affine K-algebraic sets is a homeomorphism of the underlaying topological spaces. Further, the regular (or polynomial) functions of an affine K-algebraic set $V \subseteq K^n$ are precisely the morphisms $V \to K^1$.

The canonical map

(b.1) $\quad {}^* : \operatorname{Hom}_{\mathcal{A}ff\ K\text{-}\mathcal{A}lg\,\mathcal{S}ets}(V,W) \longrightarrow \operatorname{Hom}_{K\text{-alg}}(K[W],K[V]),$

defined by

$$f \longmapsto (f^* : K[W] \to K[V],\ f^*(y_i) \mapsto f_i\,(\mathrm{mod}\ I_K(V)), i = 1,\ldots,n),$$

(it is routine to check that this map is well-defined and that f^* is a K-algebra homomorphism). The K-algebra homomorphism f^* is said to be induced from f.

Conversely, if $\varphi : K[W] \longrightarrow K[V]$ is a K-algebra homomorphism, then there are polynomials $F_1,\ldots,$ $F_n \in K[X_1,\ldots,X_m]$ such that $\varphi(y_i) = f_i := F_i\,(\mathrm{mod}\ \mathrm{I}_K(V)\,, i = 1,\ldots,n)$, and it is routine to check that these $f_1,\ldots,f_n$ define a morphism of affine K-algebraic sets $\varphi^* : V \to W$, $a \mapsto (f_1(a),\ldots,f_n(a))$

[4] A morphism $f \in \operatorname{Mor}_{\mathcal{C}}(X,Y)$ is a category $\mathcal{C}$ with objects $X, Y \in \operatorname{Obj}\mathcal{C}$ is called an isomorphism if there exists a morphism $g \in \operatorname{Mor}_{\mathcal{C}}(X,Y)$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.

which does not depend on the choice of $f_1,\ldots,f_n \in K[V]$. Altogether this defines a map

(b.2)
$$^{*} : \mathrm{Hom}_{K\text{-alg}}(K[W],K[V]) \longrightarrow \mathrm{Hom}_{\mathcal{Aff}\ K\text{-}\mathcal{Alg}}(V,W),$$
$$\varphi \longmapsto (\,\varphi^{*} : V \to W,\ a \mapsto (f_1(a),\ldots,f_n(a)) \in K^n\,).$$

which is the inverse of the map defined in **(b.1)**. This proves that there is a canonical bijection
$$\mathrm{Hom}_{\mathcal{Aff}\ K\text{-}\mathcal{Alg}\,\mathcal{Sets}}(V,W) \xrightarrow{\ \sim\ } \mathrm{Hom}_{K\text{-alg}}(K[W],K[V]).$$

(**Remark :** Note that a bijective morphism $f : V \to W$ is not necessarily an isomorphism of affine K-algebraic sets. For example, if $V := \mathrm{V}_K(XY-1) \cup \{(0,1)\} = \mathrm{V}_K(\mathfrak{a}) \subseteq K^2$, where $\mathfrak{a} := \langle XY-1\rangle \cap \langle X, Y-1\rangle$ and if $f : V \to K^1$ is the first projection $(a,b) \mapsto a$, then f is bijective, but not an isomorphism, since the K-algebra homomorphism $\varphi : K[X] = K[K^1] \to K[V] = K[X,Y]/\mathfrak{a}$, $X \mapsto X(\mathrm{mod}\,\mathfrak{a})$, associated to f is not an isomorphism.)

(i) The assignments $V \rightsquigarrow K[V]$, $f \rightsquigarrow f^*$ defines a (contravariant) functor from the category $\mathcal{Aff}\ K\text{-}\mathcal{Alg}\,\mathcal{Sets}$ of affine K-algebraic sets to the category $\mathcal{Aff}\ K\text{-}\mathcal{Alg}$ of affine K–algebras.
(**Remark :** If K is algebraically closed, then this functor is an equivalence onto the full subcategory of all reduced affine K–algebras (by the Hilbert Nullstellensatz, see R 8.1.2 (c) below.)

R 8.1.2 With the notations and definitions introduced in **R 8.1.1** above, we have :

(a) For every subset $V \subseteq K^n$, $\mathrm{V}_K(\mathrm{I}_K(V)) = \overline{V}$ (:= the closure of V in K^n with respect to the K-Zariski topology on K^n). In particular, $\mathrm{I}_K(\mathrm{V}_K(\mathrm{I}_K(V))) = \mathrm{I}_K(V)$ and $\mathrm{V}_K(\mathrm{I}_K(\mathrm{V}_K(\mathfrak{a}))) = \mathrm{V}_K(\mathfrak{a})$, where $\mathfrak{a}$ is an ideal in $K[X_1,\ldots,X_n]$. Further, the map I_K is injective. (**Hint :** Since $V \subseteq \mathrm{V}_K(\mathrm{I}_K(V))$, $\overline{V} \subseteq \mathrm{V}_K(\mathrm{I}_K(V))$. For the other inclusion, if $\mathrm{V}_K(\mathfrak{b})$ is a closed subset containing V, where $\mathfrak{b}$ is an ideal in $K[X_1,\ldots,X_n]$, then $\mathfrak{b} \subseteq \bigcap_{a\in V} \mathfrak{m}_a = \mathrm{I}_K(V)$ and hence $\mathrm{V}_K(\mathfrak{b}) \supseteq \mathrm{V}_K(\mathrm{I}_K(V))$. — **Remark :** Generally, for arbitrary field K it is rather difficult to describe the image of the map I_K in the set of radical ideals in $K[X_1,\ldots,X_n]$. However, for an algebraically closed field we have a complete answer, see the next part (b).)

(b) Let K be an *algebraically closed* field. Then the maps
$$\mathfrak{F}_K(K^n) \underset{\mathrm{V}_K}{\overset{\mathrm{I}_K}{\rightleftarrows}} \text{r-}\mathfrak{I}(K[X_1,\ldots,X_n])$$
$$V \longmapsto \mathrm{I}_K(V)$$
$$\mathrm{V}_K(\mathfrak{a}) \longleftarrow\!\shortmid\ \mathfrak{a}$$
$$\mathfrak{Irr}_K(K^n) \xleftrightarrow{\ \mathrm{I}_K\ } \mathrm{Spec}\,(K[X_1,\ldots,X_n])$$
$$\mathrm{V}_K(\mathfrak{p}) \xleftrightarrow{\ \mathrm{V}_K\ } \mathfrak{p}$$
are inclusion-reversing, bijective and mutually inverses of each other. Moreover, under this bijective correspondence irreducible affine K-algebraic sets in K^n (see R 8.0.2) corresponds to the prime ideals in $K[X_1,\ldots,X_n]$. (This is an immediate consequence of the famous geometric version of Hilbert's Nullstellensatz **(HNS 2) :** *Let K be an algebraically closed field and let $\mathfrak{a} \subseteq K[X_1,\ldots,X_n]$ be an ideal. Then $\mathfrak{I}_K(\mathrm{V}_K(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.* — **Remark :** Let K be an arbitrary field, $\mathfrak{a} \subseteq K[X_1,\ldots,X_n]$ be an ideal and let $A := K[X_1,\ldots,X_n]/\mathfrak{a}$. The ideal $\mathrm{I}_K(\mathrm{V}_K(\mathfrak{a}))/\mathfrak{a}$ in A is the intersection $\bigcap_{\xi\in K\text{-Spec}\,A} \mathfrak{m}_\xi$ of the maximal ideals $\mathfrak{m}_\xi$ in A corresponding to the points $\xi \in K\text{-Spec}\,A$ and therefore an invariant of the K-algebra A, called the K-radical $\mathfrak{r}_A$ of A. The equality $\mathrm{I}_K(\mathrm{V}_K(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ is equivalent with the condition that the nil-radical of A and the K-radical of R coincide. Therefore the equality $\mathrm{I}_K(\mathrm{V}_K(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ implies the equality $\mathrm{I}_K(\mathrm{V}_K(\mathfrak{b})) = \sqrt{\mathfrak{b}}$ for any ideal $\mathfrak{b}$ in a polynomial algebra $K[Y_1,\ldots,Y_m]$ with $A \cong K[Y_1,\ldots,Y_m]/\mathfrak{b}$.)

(c) Let K be an *algebraically closed* field and $A = K[x_1,\ldots,x_n] = K[X_1,\ldots,X_n]/\mathfrak{a}$ be a reduced affine algebra over K. Then there exists an affine K-algebraic subset $\mathrm{V} \subseteq K^n$ with the K-coordinate ring $K[V] \xrightarrow{\sim} A$ (as K-algebras). (**Hint :** Since A is reduced, $\mathfrak{a} = \sqrt{\mathfrak{a}} \in \text{r-}\mathfrak{I}(K[X_1,\ldots,X_n])$. Then $\mathrm{I}_K(\mathrm{V}_K(\mathfrak{a})) = \sqrt{\mathfrak{a}} = \mathfrak{a}$ by the geometric version HNS 2 and hence $K[V] = K[X_1,\ldots,X_n]/\mathfrak{a} \xrightarrow{\sim} A$.)

R 8.1.3 (K-Spectrum of an algebra over a field K) Let K be a field and let A be an arbitrary K-algebra. The set
$$K\text{-Spec}\,A := \{\mathfrak{m} \in \mathrm{Spm}\,A \mid A/\mathfrak{m} = K\}$$
is called the K-spectrum of A. Note that under the identifications as in R 8.1.1 (d), we have $K\text{-Spec}\,A = \mathrm{Hom}_{K\text{-alg}}(A,K)$.

(a) If $A \xrightarrow{\sim} K[X_1,\ldots,X_n]/\mathfrak{a}$ is a representation of the affine K-algebra A, then the affine K-algebraic set $\mathrm{V}_K(\mathfrak{a}) := \{a \in K^n \mid F(a) = 0 \text{ for all } F \in \mathfrak{a}\}$ is called the set of K-rational points of A. Under the bijective maps in R8.1.1 (d), we also have identifications :

$$\mathrm{V}_K(\mathfrak{a}) = \mathrm{Hom}_{K\text{-alg}}(A, K) = K\text{-Spec}\,A.$$

In particular, for every affine K-algebraic set V in K^n, we have $\mathrm{V} = \mathrm{Hom}_{K\text{-alg}}(K[\mathrm{V}], K)$.

(b) For an affine algebra A over an *algebraically closed* field K, we have $K\text{-Spec}\,A = \mathrm{Spm}\,A$.
(**Hint :** This follows from the algebraic version of Hilbert's Nullstellensatz :

HNS 3 : *For an arbitrary maximal ideal* $\mathfrak{m} \in \mathrm{Spm}\,A$ *in an affine algebra A over K, the residue field* $A/\mathfrak{m}$ *is a finite field extension of K. In particular,* $\mathrm{Spm}\,\mathbb{C}[X_1,\ldots,X_n] = \mathbb{C}\text{-Spec}\,\mathbb{C}[X_1,\ldots,X_n]$.

Note that $\mathrm{Spm}\,\mathbb{R}[X_1,\ldots,X_n] \supsetneq \mathbb{R}\text{-Spec}\,\mathbb{R}[X_1,\ldots,X_n]$ for $n \geq 1$. In fact, the maximal ideal $\mathfrak{m} := \langle X_1^2+1, X_2, \ldots X_n\rangle \notin \mathbb{R}\text{-Spec}\,\mathbb{R}[X_1,\ldots,X_n]$ and the residue field $\mathbb{R}[X_1,\ldots,X_n]/\mathfrak{m} = \mathbb{C}$ and therefore it is called a complex point of $\mathrm{Spm}\,\mathbb{R}[X_1,\ldots,X_n]$.)

(c) On the K-spectrum $K\text{-Spec}\,A$ of an arbitrary K-algebra, we define the affine K-algebraic subsets (and hence the K-Zariski topology) as follows : For $f \in A$, we put :

$$\mathrm{V}_K(f) := \{\xi \in K\text{-Spec}\,A \mid \xi(f) = 0\} = \{\xi \in K\text{-Spec}\,A \mid f(\xi) = 0\} = \{\mathfrak{m} \in K\text{-Spec}\,A \mid f \in \mathfrak{m}\}$$

and, for an ideal $\mathfrak{a}$ in A, we put $\mathrm{V}_K(\mathfrak{a}) := \bigcap_{f\in\mathfrak{a}} \mathrm{V}_K(f)$. Note that $\mathrm{V}_K(\mathfrak{a}) = \mathrm{V}_K(\sqrt{\mathfrak{a}})$.

The set $\mathfrak{F}_K(K\text{-Spec}\,A) = \{\mathrm{V}_K(\mathfrak{a}) \mid \mathfrak{a} \in \text{r-}\mathfrak{I}(A)\}$ of all affine K-algebraic sets in $K\text{-Spec}\,A$ form the closed sets of a topology — called the K-Zariski topology on $K\text{-Spec}\,A$. The open subsets are the complements $\mathrm{D}_K(\mathfrak{a}) := K\text{-Spec}\,A \smallsetminus \mathrm{V}_K(\mathfrak{a})$, $\mathfrak{a} \in \text{r-}\mathfrak{I}(A)$ and $\mathrm{D}_K(f) = \{\xi \in K\text{-Spec}\,A \mid \xi(f) \neq 0\} = K\text{-Spec}\,A \smallsetminus \mathrm{V}_K(f)$, $f \in A$ form a basis for the K-Zariski topology on $K\text{-Spec}\,A$. Therefore we have defined the inclusion reversing map :

$$\mathrm{V}_K : \text{r-}\mathfrak{I}(A) \longrightarrow \mathfrak{F}_K(K\text{-Spec}\,A), \quad \mathfrak{a} \longmapsto \mathrm{V}_K(\mathfrak{a}).$$

Further, for an arbitrary subset $V \subseteq K\text{-Spec}\,A$, we associate the radical ideal

$$\mathrm{I}_K(V) := \{f \in A \mid f(\xi) = 0 \text{ for all } \xi \in V\} = \textstyle\bigcap_{\xi\in V} \mathfrak{m}_\xi \in \text{r-}\mathfrak{I}(A).$$

These radical ideals satisfy the same properties of Exercise 8.9 (a).

(d) If A is an affine algebra over an *algebraically closed* field K, then the maps

$$\mathfrak{F}_K(K\text{-Spec}\,A) \underset{\mathrm{V}_K}{\overset{\mathrm{I}_K}{\rightleftarrows}} \text{r-}\mathfrak{I}(A)$$

$$V \longmapsto \mathrm{I}_K(V)$$

$$\mathrm{V}_K(\mathfrak{a}) \longleftarrow\!\shortmid\; \mathfrak{a}$$

are inclusion-reversing, bijective and mutually inverses of each other.

(e) Let K be an arbitrary field. Then for every K-algebra homomorphism $\varphi : A \to B$ of arbitrary K-algebras, we define the map $\varphi^* : K\text{-Spec}\,B \to K\text{-Spec}\,A$ by $\varphi^*\xi := \xi \circ \varphi$ or by $\varphi^*\mathfrak{m} = \varphi^{-1}(\mathfrak{m})$, $\mathfrak{m} = \mathrm{Ker}\,\xi \in K\text{-Spec}\,B = \mathrm{Hom}_{K\text{-alg}}(B, K)$. Then $(\mathrm{id}_A)^* = \mathrm{id}_{K\text{-Spec}\,A}$ and further, if $\psi : B \to C$ is an another K-algebra homomorphism then $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.
Therefore, the assignments $A \rightsquigarrow K\text{-Spec}\,A$, $\varphi \rightsquigarrow \varphi^*$ define a contravariant functor from the category $\mathcal{A}\!f\!f\,K\text{-}\mathcal{A}lgs$ of K-algebras to the category $\mathcal{T}ops$ of topological spaces. (For a K-algebra homomorphism $\varphi : A \to B$ the continuity of $\varphi^* : K\text{-Spec}\,B \to K\text{-Spec}\,A$, $\xi \mapsto \xi\varphi$, is immediate from the more precise assertion : *For a K-algebra homomorphism* $\varphi : A \to B$ *and an ideal* $\mathfrak{a}$ *in A, we have* $(\varphi^*)^{-1}(\mathrm{V}_K(\mathfrak{a})) = \mathrm{V}_K(\mathfrak{a}B)$.)

R 8.1.4 (Algebra-Geometry Lexion) We give a brief summary of the algebra-geometry lexicon. Let K be an *algebraically closed* [5] field, $V \in \mathfrak{F}(K^n)$ be a (fixed) Zariski closed subset in K^n, $\mathfrak{F}_K(V)$ be the set of all Zariski closed subsets of V (and hence also closed in K^n) and let $K[V]$ be the K-coordinate ring of V.

(a) For an ideal $\mathfrak{a} \in \mathfrak{I}(K[V])$, we put $\mathrm{V}_{K,V}(\mathfrak{a}) := \{a \in V \mid f(a) = 0 \text{ for all } f \in \mathfrak{a}\}$ which is a common zero set of all functions $f \in \mathfrak{a}$ in V.
Further, for a subset $W \subseteq V$, we put $\mathrm{I}_{K[V]}(W) := \{f \in K[V] \mid f(a) = 0 \text{ for all } a \in W\}$ which is the set of all functions $f \in K[V]$ vanishing on W. Clearly, $\mathrm{I}_{K[V]}(W) = \bigcap_{a\in W} \mathfrak{m}_a \in \text{r-}\mathfrak{I}(K[V])$.

[5] We should mention that some parts of this lexion stay intact even if we drop the hypothesis that K is algebraically closed

Then the maps

$$\begin{array}{ccc} \mathcal{F}_K(V) & \underset{\mathrm{V}_{K,V}}{\overset{\mathrm{I}_{K[V]}}{\rightleftarrows}} & \text{r-}\mathcal{I}(K[V]) \\ W & \longmapsto & \mathrm{I}_{K[V]}(W) \\ \mathrm{V}_{K,V}(\mathfrak{a}) & \longleftarrow\!\shortmid & \mathfrak{a} \end{array} \qquad \textbf{(a.1)}$$

$$\begin{array}{ccc} \mathcal{F}\mathrm{rr}_K(V) & \underset{\mathrm{V}_{K,V}}{\overset{\mathrm{I}_{K[V]}}{\longleftrightarrow}} & \mathrm{Spec}\,(K[V]) \\ \mathrm{V}_{K,V}(\mathfrak{p}) & \longleftrightarrow & \mathfrak{p} \end{array} \qquad \textbf{(a.2)}$$

$$\begin{array}{ccc} V & \underset{\mathrm{V}_{K,V}}{\overset{\mathrm{I}_{K[V]}}{\longleftrightarrow}} & K\text{-Spec}\,(K[V]) \\ a & \longleftrightarrow & \mathfrak{m}_a \end{array} \qquad \textbf{(a.3)}$$

$$\begin{array}{ccc} \mathcal{F}\mathrm{rr}\,\mathcal{C}\mathrm{om}_K(V) & \underset{\mathrm{V}_{K,V}}{\overset{\mathrm{I}_{K[V]}}{\longleftrightarrow}} & \mathrm{Min\,Spec}\,(K[V]) \\ \mathrm{V}_{K,V}(\mathfrak{p}) & \longleftrightarrow & \mathfrak{p} \end{array}$$

$\mathrm{I}_{K[V]}$ and $\mathrm{V}_{K,V}$ are inclusion-reversing, bijective and mutually inverses of each other. Moreover, under this bijective correspondence irreducible Zariski closed sets in V (see R 8.0.2) corresponds to the prime ideals in $K[V]$ and (points of) V corresponds to the K-Spec $K[V]$. In particular, V is irreducible if and only if $K[V]$ is an affine domain. Furthermore, irreducible components of V corresponds to the minimal prime ideals in $K[V]$ (see Exercise 8.10).

(b) For two affine K-algebraic sets $V \subseteq K^m$ and $W \subseteq K^n$, there is a bijective corrspondence :

$$\mathrm{Hom}_{\mathcal{A}\!f\!f\; K\text{-}\mathcal{A}lg\,\mathcal{S}ets}(V,W) \xleftrightarrow{\ \sim\ } \mathrm{Hom}_{K\text{-alg}}(K[W],K[V]),$$

Under this correspondence isomorphisms are mapped bijectively onto isomorphisms, but behaves less well with respect to injectivity (see Exercise 8.26). The composition of two morphisms of affine K-algebraic sets corresponds to the composition of K-algebra homomorphisms of the coordinate rings, but in the reversed order.

R 8.2 The Prime Spectrum of a (commutative) ring and the Zariski Topology. There is no affine K-algebraic set (K some field) associated to a general commutative ring A. The abstract substitute for an affine K-algebraic set is the prime spectrum Spec A. Since (see R 8.1.4 (a.3)) affine K-algebraic sets over algebraically closed field K are embedded into the prime spectrum of its coordinate ring. We therefore can regard the prime spectrum of a commutative ring as a generalization of an affine K-algebraic set. Statements about spectra of rings always imply statements about affine K-algebraic sets as special cases. As in the R 8.1, we will summarize some algebra-geometry correspondences.

Let A be a commutative ring and let Spec A (resp. Spm A) denote the set of all prime ideals (resp. maximal ideals) in A ; — called the prime spectrum (resp. the maximal spectrum) of the ring A.

R 8.2.1 Notation. For the purposes of geometry, $X = \mathrm{Spec}\,A$ is viewed as point set. A point $x \in X$ is a prime ideal in A and we shall also denote it by the ideal-like notation $\mathfrak{p}_x$ in situations where we want to consider it as the prime ideal in A. If $x \in \mathrm{Spm}\,A \subseteq X$, we denote it also by $\mathfrak{m}_x$. The residue field $\kappa(x) := A_{\mathfrak{p}_x}/\mathfrak{p}_x A_{\mathfrak{p}_x} = \mathrm{Q}(A/\mathfrak{p}_x)$ of the local ring $A_{\mathfrak{p}_x}$ (which is also the quotient field of the integral domain $A/\mathfrak{p}_x$) is called the field of the point x. For a K-algebra the K-spectrum K-Spec A is the subset $\{x \in X \mid \kappa(x) = K\} \subseteq X$. The field $\kappa(x)$ of the point $x \in X$ is related to the ring A via the canonical ring homomorphisms $A \to A/\mathfrak{p}_x \hookrightarrow \kappa(x)$. For an element $f \in A$ and $x \in X$, $f(x)$, we denote the image of f in $\kappa(x)$ by $f(x)$ and call it the value of f at the point x. This extends the analogous notation for the K-spectrum, see also Exercise 7.8. But now the function $x \longmapsto f(x)$, $x \in X$, has, in general, values in different fields. A point $x \in X$ is a zero of $f \in A$ if and only if $f \in \mathfrak{p}_x$. Note that an equation $f(x) = 0$ for a function $f \in A$ and $x \in X$ is equivalent to $f \in \mathfrak{p}_x$ and an equation $(fg)(x) = 0$ for two functions $f, g \in A$ and a point $x \in X$ is equivalent to either $f(x) = 0$ or $g(x) = 0$. The function $x \longmapsto f(x)$ is identically zero on X if and only if $f \in \bigcap_{x \in X} \mathfrak{p}_x = \mathfrak{n}_A = \sqrt{0}$. Therefore, if A is not reduced, i. e. if A has non-trivial nilpotent elements, then the function $x \longmapsto f(x)$ can be identically zero without f being zero element of A. The set of all elements $f \in A$ which vanish on $\operatorname{Spm} A$ is the Jacobson radical $\mathfrak{m}_A = \bigcap_{x\in \operatorname{Spm} A} \mathfrak{m}_x$ of A, and for K-algebra A, the set of all $f \in A$ which vanish on K-$\operatorname{Spec} A$ is the K-radical $\mathfrak{r}_A = \bigcap_{x\in K\text{-}\operatorname{Spec} A} \mathfrak{m}_x$ of A.

R 8.2.2 Affine Algebraic sets and Zariski topology. Let A be a (commutative) ring. For an ideal $\mathfrak{a}$ in A, the set of common zeros

$$\mathrm{V}(\mathfrak{a}) := \{x \in \operatorname{Spec} A \mid f(x) = 0 \text{ for all } f \in E\} = \textstyle\bigcap_{f\in\mathfrak{a}} \mathrm{V}(f) = \{x \in \operatorname{Spec} A \mid \mathfrak{a} \subseteq \mathfrak{p}_x\}$$

of all elements $f \in \mathfrak{a}$ in $X = \operatorname{Spec} A$, is called an *(affine) algebraic set* in $\operatorname{Spec} A$ defined by $\mathfrak{a}$. Note that $\mathrm{V}(\mathfrak{a}) = \mathrm{V}(\sqrt{\mathfrak{a}})$. Further, the set $\mathcal{F}(\operatorname{Spec} A) := \{\mathrm{V}(\mathfrak{a}) \mid \text{r-}\mathfrak{I}(A)\}$ of all affine algebraic sets in $\operatorname{Spec} A$ form the closed sets for a topology —called the *Zariski topology* on $\operatorname{Spec} A$. The open subsets are the complements $\mathrm{D}(\mathfrak{a}) : \operatorname{Spec} \smallsetminus \mathrm{V}(\mathfrak{a})$, $\mathfrak{a} \in \text{r-}1\mathfrak{I}(A)$ and $\mathrm{D}(f) = \operatorname{Spec} A \smallsetminus \mathrm{V}(f) = \{x \in \operatorname{Spec} A \mid f(x) \neq 0\}$, $f \in A$, form a basis for the Zariski topology on $\operatorname{Spec} A$. The basic open set $\mathrm{D}(f)$ is also known as the *domain* of f which explains the usage of the letter D.

Therefore we have defined the inclusion reversing map :

$$\mathrm{V} : \text{r-}\mathfrak{I}(A) \longrightarrow \mathcal{F}_K(\operatorname{Spec} A), \quad \mathfrak{a} \longmapsto \mathrm{V}(\mathfrak{a}).$$

For a better understanding of the map V_K, we define the inclusion reversing map in the opposite direction :

$$\mathrm{I} : \mathcal{F}_K(\operatorname{Spec} A) \longrightarrow \text{r-}\mathfrak{I}(A).$$

For this to *every* subset $Y \subseteq X = \operatorname{Spec} A$, we associate the radical ideal (in A)

$$\mathrm{I}_K(Y) := \{f \in A \mid f(y) = 0 \text{ for all } y \in Y\} = \textstyle\bigcap_{y\in Y} \mathfrak{p}_y \in \text{r-}\mathfrak{I}(A)$$

is called the *ideal of* Y. Note that $f \in \mathfrak{p}_y$ if and only if $f(y) = 0$. This implies that $\mathrm{I}(\{y\}) = \mathfrak{p}_y$ for all $y \in X$. Further, the reduced ring

$$A(Y) := A/\mathrm{I}(Y)$$

is called the *coordinate ring of* Y. is a K-algebra isomorphism.

R 8.2.3 Formal Hilbert's Nullstellensatz. With the notations and definitions introduced as in ?? above, let A be a ring and let $X = \operatorname{Spec} A$ be the prime spectrum of A (with Zariski topology). Then :

(a) For every subset $Y \subseteq X = \operatorname{Spec} A$, $\mathrm{V}(\mathrm{I}(Y)) = \overline{Y}$(:= the closure of Y in X with respect to the Zariski topology). In particular, $\mathrm{I}(\mathrm{V}(\mathrm{I}(Y))) = \mathrm{I}(Y)$ and $\mathrm{V}(\mathrm{I}(\mathrm{V}(\mathfrak{a}))) = \mathrm{V}(\mathfrak{a})$, where $\mathfrak{a} \in \mathfrak{I}(A)$.

(b) (*Formal Hilbert's Nullstellensatz*[6]) $\mathrm{I}(\mathrm{V}(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ for every ideal $\mathfrak{a} \in \mathfrak{I}(A)$.

(c) The maps

$$\begin{array}{ccc} \mathcal{F}(X) & \underset{\mathrm{V}}{\overset{\mathrm{I}}{\rightleftarrows}} & \text{r-}\mathfrak{I}(A) \\ Y & \longmapsto & \mathrm{I}(Y) \\ \mathrm{V}(\mathfrak{a}) & \longleftarrow\!\shortmid & \mathfrak{a} \end{array}$$

are inclusion-reversing, bijective and mutually inverses to each other. (**Remark :** In the special case that if $A = K[V]$ is the K-coordinate ring of affine K-algebraic set V over an algebraically closed field K the correspondence in the above part (c)is a generalization of the correspondence in R 8.1.4 (a).)

(d) (*Functorial Properties of Spec*) A ring homomorphism $\varphi : A \to B$ induces a morphism $\varphi^* : \operatorname{Spec} B \to \operatorname{Spec} A$, $\mathfrak{q} \mapsto \varphi^{-1}(\mathfrak{q})$, on spectra. Note that the correspondence between ring homomorphisms and morphisms of spectra is not bijective. See Exercise 8.22 for more on functorial properties. (**Remark :** In general, φ^* does not restrict to a map $\operatorname{Spm} B \to \operatorname{Spm} A$, but if φ is a K-algebra homomorphism of affine K-algebras, then it does. If in addition $A = K[W]$ and $B = K[V]$ are coordinate rings of affine K-algebraic sets over algebraically closed field K, then the correspondence in R 8.1.4 (a) translate this restriction of φ^* in to a map $V \to W$ which is exactly the morphism corresponding to φ.)

R 8.2.4 Example. For the zero ring 0 the prime spectrum $\operatorname{Spec} 0 = \emptyset$. For a principal ideal domain A, for example, $A = \mathbb{Z}$, or $A = K[X]$ or $A = K[\![X]\!]$, where K is a field and X is an indeterminate over K. The prime spectrum $X = \operatorname{Spec} A$ consists of the zero (prime) ideal $0 \subseteq A$ and of all pricipal ideals $\langle p \rangle \subseteq A$ generated by by prime

[6] Also known as *Scheinnullstellensatz*

elements $p \in A$, i. e.

$$\operatorname{Spec} A = \{0, \langle p \rangle \mid p \in \mathbb{P}(A)\},$$

where $\mathbb{P}(A)$ denote the complete representative set for prime elements in A under the equivalence relation of being associates in A. In particular,

$$\operatorname{Spec} \mathbb{Z} = \{0\} \cup \mathbb{P}, \ \operatorname{Spec} K[X] = \{0\} \cup \{\pi \in \mathbb{P}(K[X])\} \ \text{ and } \operatorname{Spec} K[\![X]\!] = \{0, \langle X \rangle\},$$

where $\mathbb{P}(K[X]) = \{\pi \in K[X] \mid \pi \text{ non-constant monic irreducible over } K\}$.

Furthermore, the closed subsets in X are of type $\mathrm{V}(a)$, $a \in A$. In particular, $\mathrm{V}(a) = X$ for $a = 0$ and $\mathrm{V}(a) = \emptyset$ if $a \in A^\times$ is a unit in A. For elements $a \in A \smallsetminus (A^\times \cup \{0\})$, $\mathrm{V}(a) = \{\langle p_1 \rangle, \ldots, \langle p_r \rangle\}$, where $a = \varepsilon\, p_1^{\nu_1} \cdots p_r^{\nu_r}$ is a prime factorization with pairwise coprime prime factors $p_1, \ldots, p_r$, exponents $\nu_1, \ldots, \nu_r > 0$ and a unit $\varepsilon \in A^\times$. In particular, all prime ideals which are generated by prime elements or, equivalently, all non-zero prime ideal in A gives rise to closed points in $X = \operatorname{Spec} A$. Further, a subset $V \subseteq X$ is closed if and only if it coincides with X or $\emptyset$, or if it is a finite set of closed points. Therefore the zero ideal $0 \subseteq A$ yields a dense point in X, i. e. $\overline{\{0\}} = X$. In particular, if A is not a field, then $X \neq \{0\}$ and the point $0 \in X$ cannot be closed.

Switching to complements, the open subsets in X are $\emptyset$, X and the sets of type $X \smallsetminus \{x_1, \ldots, x_r\}$, where $x_1, \ldots, x_r \in X$ are finitely many closed points. Therefore any non-empty open subset of X will contain the point given by the zero ideal $0 \subseteq A$ and hence the Zariski topology on $X = \operatorname{Spec} A$ cannot satisfy the Hausdorff separation axiom, unless A is a field.

R 8.2.5 It follows from the formal Hilbert's nullstellensatz (R 8.23 (b)) that the prime spectrum $\operatorname{Spec} A$ of a ring A is a noetherain topological space if and only if the radical ideals in A satisfy the ascending chain condition, in particular, if A is a noetherain ring, then the prime ideals satisfy ACC and $X = \operatorname{Spec} A$ is a noetherian topological space. More generally, if $A = \sum_{i=1}^n A f_i$ is a finitely generated ideal in a ring A, then $\mathrm{D}(\mathfrak{a}) = \cup_{i=1}^n \mathrm{D}(f_i)$ is quasi-compact.

The ring $\mathbb{Q}[X_i \mid i \in \mathbb{N}] / \langle X_i^2 \mid i \in \mathbb{N} \rangle$ is not noetherain, but its prime spectrum $\{\mathfrak{m} := \langle x_i \mid i \in \mathbb{N} \rangle\}$ is singleton and hence noetherian.

8.1 Let X be a topological space and $Y \subseteq X$ be an irreducible subspace.

(a) Suppose that $Y = \cup_{i=1}^n Y_i$ with each Y_i , is closed in Y. Then $Y = Y_i$ for some $i = 1, \ldots, n$.

(b) Suppose that $Y \subseteq \cup_{i=1}^n X_i$ with each X_i is closed in X. Then $Y \subseteq X_i$ for some $i = 1, \ldots, n$.

(c) The closure $\overline{Y}$ of Y is also irreducible.

(d) Y is contained in a maximal irreducible subspace. (Maximal elements in the ordered set $(\mathfrak{Irr}(X), \subseteq)$ (where $\mathfrak{Irr}(X)$ is the set of all irreducible subsets X) are called *maximal irreducible subspace*. — **Hint:** Let $\mathcal{S} := \{Z \in \mathfrak{Irr}(X) \mid Y \subseteq Z\}$. Then $Y \in \mathcal{S}$ and $\mathcal{S}$ is ordered by the natural inclusion $\subseteq$. Further for a chain (totally ordered subset) $\mathcal{C}$ in $\mathcal{S}$, $Z' := \cup_{Z \in \mathcal{C}}$ is irreducible in X, since for non-empty open subsets U, V of Z', there exists $Z \in \mathcal{C}$ with $U \cap Z \neq \emptyset$ and $V \cap Z \neq \emptyset$, hence $(U \cap Z) \cap (V \cap Z) \neq \emptyset$ (since Z is irreducible). This proves that $Z' \in \mathcal{S}$ is an upper bound for $\mathcal{C}$ in $\mathcal{S}$ and hence Zorn's Lemma yields (d). In particular, $\operatorname{Max}(\mathfrak{Irr}(X), \subseteq) \neq \emptyset$. Its elements are called *irreducible components* of X.)

(e) The maximal irreducible subspaces of X are closed and cover X.

(f) Let $\psi : X \to X'$ be a continuous map between topological spaces. Show that if $V \subseteq X$ is an irreducible subset, then its image $\psi(V)$ and its closure $\overline{\psi(V)}$ are irreducible in X'.

8.2 Let X be a *noetherian* topological space. Then :

(a) Every closed subset $V \subseteq X$ is a finite union $V = V_1 \cup \cdots \cup V_r$ of irreducible components $V_1, \ldots, V_r$ of V and $V_i \not\subseteq \cup_{j \neq i} V_j$ for every $1 \leq i \leq r$. In particular, X has only finitely many irreducible components. (**Hint:** Consider the collection $\mathcal{S}$ of those closed subsets Z of X which cannot be expressed as a union of finitely many closed irreducible subsets in X.)

(b) X has only finitely many connected components and every connected component is a union of some irreducible components of X. In particular, the connected components of X are (closed and) open.

8.3 The following three special cases of the maps associated to K-algebra homomorphisms on the K-spectra which are important and are used often, see R 8.1.3 (e).

Let A be an arbitrary K-algebra over a field K.

(a) Let $\varphi : A \to A'$ be a *surjective* K-algebra homomorphism. Then the map associated to φ on K-Spectra $\varphi^* : K\text{-Spec}\, A' \to K\text{-Spec}\, A$ is a continuous closed embedding with $\text{Img}\, \varphi^* = \text{V}_K(\text{Ker}\, \varphi)$. In particular, *the map* $\pi_{\mathfrak{a}}^* : K\text{-Spec}\,(A/\mathfrak{a}) \to K\text{-Spec}\, A$, *associated to the residue-class homomorphism* $\pi_{\mathfrak{a}} : A \to A/\mathfrak{a}$ *induces a (closed embedding) homeomorphism* $K\text{-Spec}\,(A/\mathfrak{a}) \xrightarrow{\sim} \text{V}_K(\mathfrak{a}) \subseteq K\text{-Spec}\, A$.

(b) *The residue-class homomorphism* $\pi : A \to A/\mathfrak{n}_A$, *where* $\mathfrak{n}_A$ *is the nil-radical of* A, *induces a homeomorphism* $\pi^* : K\text{-Spec}\,(A/\mathfrak{n}_A) \xrightarrow{\sim} K\text{-Spec}\, A$. (**Hint:** This is a special case of (a), since $\text{V}_K(\mathfrak{n}_A) = K\text{-Spec}\, A$.)

(c) *For* $f \in A$, *the canonical* K*-algebra homomorphism* $\iota_f : A \to A_f = A[1/f]$ *induces (an open embedding) a homeomorphism* $\iota_f^* : K\text{-Spec}\, A_f \xrightarrow{\sim} \text{D}_K(f) \subseteq K\text{-Spec}\, A$.
More generally, for an arbitrary multiplicatively closed subset $S \subseteq A$, the canonical homomorphism $\iota_S : A \to A^{-1}S$ induces a homeomorphism from $K\text{-Spec}\, S^{-1}A \xrightarrow{\sim} \text{Img}\, \iota_S^* = \{\mathfrak{m} \in K\text{-Spec}\, A \mid \mathfrak{m} \cap S = \emptyset\} \subseteq K\text{-Spec}\, A$.

8.4 Let K be a field and A be a K-algebra.

(a) K-Spec A is dense in Spec$A$ if and only if the nilradical $\mathfrak{n}_A$ of A coincides with the K-radical $\mathfrak{r}_A = \bigcap_{\xi \in K\text{-Spec}\, A} \mathfrak{m}_\xi$ of A.

(b) The closed irreducible subsets in K-Spec A are precisely the sets $\text{V}_K(\mathfrak{p})$, where $\mathfrak{p} \in \text{Spec}\, R$ is a prime ideal with $\mathfrak{I}_K(\text{V}_K(\mathfrak{p})) = \mathfrak{p}$.

8.5 Let $X = \text{Spec}A$ be the prime spectrum (with Zariski topology) of a ring A.

(a) For every $x \in X$, the closure $\overline{\{x\}} = \text{V}(\mathfrak{p}_x) = \{y \in X \mid \mathfrak{p}_x \subseteq \mathfrak{p}_y\}$.

(b) A point $x \in X$ is closed, i. e. $\{x\}$ is closed in X if and only if the prime ideal $\mathfrak{p}_x$ corresponding to x is a maximal ideal in A.

(c) Spm A is dense in Spec$A$ if and only if $\mathfrak{n}_A = \mathfrak{m}_A$, where $\mathfrak{n}_A$ and $\mathfrak{m}_A$ denote the nil-radical and the Jacobson radical ideal of A, respectively.

8.6 Let $X = \text{Spec}A$ be the prime spectrum of the ring A. The Zariski topology on X does not necessarily satisfy the Hausdorff (also known as T_2) separation axiom, see Example R 8.2.4, however the following weaker separation axiom (known as T_0) holds:

(a) The Zariski topology on $X = \text{Spec}A$ yields a Kolmogrov space, i. e. a topological space satisfying the following separation axiom T_0: Given any two distinct points $x, x' \in X$, $x \neq x'$, there exists an open neighbourhood U of x such that $x' \notin U$, *or* an open neighbourhood of U' of x' such that $x \notin U'$.

(b) For functions $f, f' \in A$, the following statements are equivalent:

(i) $\text{D}(f) = \text{D}(f')$. (ii) $\text{V}(f) = \text{V}(f')$. (iii) $\text{rad}(f) = \text{rad}(f')$.

(c) $X = \text{Spec}A$ is a Hausdorff if and only if every prime ideal in A is maximal, i. e. $\dim A \leq 0$. If Spec$A$ is a Hausdorff space, then Spec$A$ is compact and *totally disconnected*, i. e. the only connected subsets are singletons. (Recall that dim$A$ denote the *Krull-dimension* of

A, see Exercise Set 10. The implication " $\dim A = 0 \Rightarrow \operatorname{Spec} A$ is Hausdorff " is not obvious. — **Hint :** Note that A is reduced with $\dim A \leq 0$ if and only if every principal ideal (or every finitely generated) ideal is generated by an idempotemnt element.)

8.7 Let A be a ring. Then :

(a) For every $g \in A$, the subset $\mathrm{D}(g) \subseteq X$ is quasi-compact (with respect to the induced Zariski topology on X). In particular, $X = \mathrm{D}(1)$ is quasi-compact.

(b) An open subset $U \subseteq \operatorname{Spec} A$ is quasi-compact if and only if $\operatorname{Spec} A \smallsetminus U = V(\mathfrak{a})$ for some finitely generated ideal $\mathfrak{a} \subseteq A$.

8.8 Let A be a ring.

(a) Let $\mathfrak{a} \subseteq A$ be an ideal and let $\pi_{\mathfrak{a}} : A \to A/\mathfrak{a}$ be the canonical residue-class homomorphism. Then the map associated to $\pi_{\mathfrak{a}}$ on spectra

$$\pi_{\mathfrak{a}}^* : \operatorname{Spec} A/\mathfrak{a} \longrightarrow \operatorname{Spec} A, \ \mathfrak{p} \longmapsto \pi_{\mathfrak{a}}^{-1}(\mathfrak{p}),$$

induces a homeomorphism of topological spaces $\operatorname{Spec} A/\mathfrak{a} \xrightarrow{\sim} \mathrm{V}(\mathfrak{a})$, where $\mathrm{V}(\mathfrak{a})$ is equipped with the subspace topology induced by the Zariski topology of $\operatorname{Spec} A$.

(b) The canonical residue-class homomorphism $\pi_{\mathfrak{n}_A} : A \longrightarrow A/\mathfrak{n}_A =: A_{\mathrm{red}}$ induces a canonical homeomorphism $\pi_{\mathfrak{n}_A}^* : \operatorname{Spec} A_{\mathrm{red}} \xrightarrow{\sim} \operatorname{Spec} A$, where $\mathfrak{n}_A$ is the nil-radical of A.

8.9 Let $X = \operatorname{Spec} A$ be the prime spectrum of a ring A and let $\mathfrak{n}_A$ be the nil-radical of A.

(a) The following statements are equivalent :

(i) X is an irreducible topological space with respect to the Zariski topology.
(ii) $A_{\mathrm{red}} := A/\mathfrak{n}_A$ is an integral domain.
(iii) $\mathfrak{n}_A$ is a prime ideal in A.

(b) Let $Y \subseteq X$ be a closed subset. Then Y is irreducible if and only if $\mathrm{I}(Y)$ is a prime ideal in A.

(c) Let $\mathfrak{Irr}(X)$ be the set of all irreducible closed subsets in X. The maps

$$\mathfrak{Irr}(X) \underset{\mathrm{V}}{\overset{\mathrm{I}}{\rightleftarrows}} \operatorname{Spec} A$$

$$Y \longmapsto \mathrm{I}(Y)$$
$$\overline{\{y\}} = \mathrm{V}(\mathfrak{p}_y) \longleftarrow\!\!\shortmid \ \mathfrak{p}_y$$

are inclusion-reversing, bijective and mutually inverses to each other. (In particular, every irreducible closed subset $Y \subseteq X$ contains a unique point $y \in$ such that $Y = \overline{\{y\}}$. This unique point is called the g e n e r i c p o i n t of Y and the points of its closure $\overline{\{y\}}$ are called *specializations* of y. More precisely, a point $x \in X$ is called a s p e c i a l i z a t i o n of a point $y \in X$ if $x \in \overline{\{y\}}$, or equivalently, $\mathfrak{p}_y \subseteq \mathfrak{p}_x$. For example, if A is an integral domain, then $\operatorname{Spec} A$ is irreducible and hence admits a unique generic points which corresponds to the zero prime ideal.)

8.10 (M i n i m a l P r i m e I d e a l s) Let A be a ring.

(a) Let $S \subseteq A$ be a multiplicatively closed subset in A and $\mathfrak{a} \subseteq A$ be an ideal in A with $\mathfrak{a} \cap S = \emptyset$. Then the ordered set $\mathcal{M} := \{\mathfrak{b} \in \mathfrak{I}(A) \mid \mathfrak{a} \subseteq \mathfrak{b} \subseteq A \smallsetminus S\}$ (with respect to the natural inclusion $\subseteq$) has maximal elements. Moreover, every such maximal element in $\mathcal{M}$ is a prime ideal in A. In particular, if $A \neq 0$, then $\operatorname{Spec} A \neq \emptyset$.

(b) The set $\mathrm{ZDiv}(A)$ of all zerodivisors in A is a union of (some) prime ideals in A. In particular, the set $S_0 = \mathrm{Nzd}(A)$ of non-zerodivisors in A is a saturated multiplicatively closed subset in A, see also Exercises 7.2 and Exercise 7.3.

(c) Suppose that $A \neq 0$. Then the ordered set $(\mathrm{Spec}\,A, \subseteq)$ has minimal elements — called minimial prime ideals in A and every $\mathfrak{q} \in \mathrm{Spec}\,A$ contains a minimal prime ideal. *Prove that every minimal prime ideal $\mathfrak{p}$ in A is contained in the set of zerodivisors in A.*

(**Proof:** Note that $\mathrm{Spec}\,A_{\mathfrak{P}} = \{\mathfrak{p}A_{\mathfrak{p}}\}$, since $\mathfrak{p}$ is a minimal element in $(\mathrm{Spec}\,A, \subseteq)$. Therefore $\mathfrak{n}_{A_{\mathfrak{p}}} = \mathfrak{p}A_{\mathfrak{p}}$ and so for every $a \in \mathfrak{p}$, $a/1 \in \mathfrak{p}A_{\mathfrak{p}}$ is nilpotent in $A_{\mathfrak{p}}$, i. e. $sa^n = 0$ for some $s \in A \smallsetminus \mathfrak{p}$ and for some minimal $n \geq 1$. Then $sa^{n-1} \neq 0$, but $(sa^{n-1})a = 0$ and hence a is a zerodivisor in A.)

8.11 (a) Show that a *noetherian* ring A has only *finitely* many minimal prime ideals, i. e. the set of minimal elements in the ordered set $(\mathrm{Spec}\,A, \subseteq)$ is $\mathrm{Min}\,(\mathrm{Spec}\,A, \subseteq) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ finite (and non-empty if $A \neq 0$). The irreducible components of $\mathrm{Spec}\,A$ are $\mathrm{V}(\mathfrak{p}_1), \ldots, \mathrm{V}(\mathfrak{p}_r)$.
(**Remark:** Note that even if A is not noetherian the ordered set $(\mathrm{Spec}\,A, \subseteq)$ has minimal elements, See above Exercise 8.10 (c). However, there are rings with infinitely many prime ideals, for example, in Boolean rings!)

(b) Let A be a ring and $K = \mathrm{Q}(A)$ be the total quotient ring of A. Further, let $\mathcal{M}(A) := \mathrm{Min}\,(\mathrm{Spec}\,A, \subseteq)$ be the set of all minimal prime ideals in A. Suppose that $\mathcal{M}(A)$ is finite (for example, if A in noetherian). Show that the following statements are equivalent:
(i) A is reduced.
(ii) $\mathrm{ZDiv}\,(A) = \bigcup_{\mathfrak{p}\in\mathcal{M}(A)} \mathfrak{p}$ and $A_{\mathfrak{p}} = \mathrm{Q}(A/\mathfrak{p})$ (the quotient field of $A/\mathfrak{p}$) for each $\mathfrak{p} \in \mathcal{M}(A)$.
(iii) $K/\mathfrak{p}K = \mathrm{Q}(A/\mathfrak{p})$ (the quotient field of $A/\mathfrak{p}$) for each $\mathfrak{p} \in \mathcal{M}(A)$ and $K = \prod_{\mathfrak{p}\in\mathcal{M}(A)} K/\mathfrak{p}K$.

(**Proof:** The implication (i)⇒(ii) also holds *without the assumption that the set $\mathcal{M}(A)$ is finite.* For a proof, first note that $\cup_{\mathfrak{p}\in\mathcal{M}(A)}\mathfrak{p} \subseteq \mathrm{ZDiv}\,(A)$ by Exercise 8.10 (c). Conversely, if $ab = 0$ with $a \in A$, $a \neq 0$, $b \in A$ and if $a \notin \mathfrak{p}$ for some $\mathfrak{p} \in \mathcal{M}(A) := \mathrm{Min}(\mathrm{Spec}\,A, \subseteq)$. Then $b \in \mathfrak{p}$. Therefore, if $a \notin \cup_{\mathfrak{p}\in\mathcal{M}(A)}\mathfrak{p}$, then $b \in \cap_{\mathfrak{p}\in\mathcal{M}(A)}\mathfrak{p} = \mathfrak{n}_A = 0$, since A is reduced by assumption. This proves that if $a \notin \cup_{\mathfrak{p}\in\mathcal{M}(A)}\mathfrak{p}$, then $a \notin \mathrm{ZDiv}(A)$ and hence $\mathrm{ZDiv}(A) \subseteq \cup_{\mathfrak{p}\in\mathcal{M}(A)}\mathfrak{p}$. Therefore the equality $\mathrm{ZDiv}(A) = \cup_{\mathfrak{p}\in\mathcal{M}(A)}\mathfrak{p}$. Let $\mathfrak{p} \in \mathcal{M}(A)$ be fixed. then $A_{\mathfrak{p}}$ is reduced (since A is reduced by assumption (i) and Exercise 7.26 (a)) and $\mathrm{Spec}\,A_{\mathfrak{p}} = \{\mathfrak{p}A_{\mathfrak{p}}\}$ and hence $A_{\mathfrak{p}}$ is a field. Further, $A_{\mathfrak{p}} == A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = \mathrm{Q}(A/\mathfrak{p})$.

(iii)⇒ (i): Since $\mathcal{M}(A)$ is finite, K is a finite product of fields and hence K is reduced. Further, the canonical ring homomorphism $A \to K$, $a \mapsto (\overline{a}_{\mathfrak{p}})_{\mathfrak{p}\in\mathcal{M}(A)}$ (where $\overline{a}_{\mathfrak{p}}$ denote the residue class of a in $A/\mathfrak{p} \subseteq \mathrm{Q}(A/\mathfrak{p})$) is injective and hence A reduced.

(ii)⇒(iii): Put $S := A \smallsetminus \mathrm{ZDiv}\,(A)$ and let $\mathfrak{q} \in \mathrm{Spec}\,A$ with $\mathfrak{q} \cap S = \emptyset$. Then $\mathfrak{q} \subseteq \mathrm{ZDiv}\,(A) = \cup_{\mathfrak{p}\in\mathcal{M}(A)}\mathfrak{p}$ by assumption (ii) and hence, since $\mathcal{M}(A)$ is finite, by Prime avoidance Lemma (see Exercise 1.9 (b)) $\mathfrak{q} \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \mathcal{M}(A)$. Therefore $\mathfrak{q} = \mathfrak{p}$, since $\mathfrak{p}$ is minimal. But $K = \mathrm{Q}(A) = S^{-1}A$. Therefore $\mathrm{Spec}\,K = \{\mathfrak{p}K \mid \mathfrak{p} \in \mathcal{M}(A)\} = \mathcal{M}(K) = \mathrm{Spm}\,K$. Further, for a fixed $\mathfrak{p} \in \mathcal{M}(A)$, $K/\mathfrak{p}K = S^{-1}A/\mathfrak{p}S^{-1}A = S^{-1}(A/\mathfrak{p})$ and hence $S^{-1}(A/\mathfrak{p})$ is a field. But, clearly $S^{-1}(A/\mathfrak{p}) \subseteq \mathrm{Q}(A/\mathfrak{p})$ and hence $K/\mathfrak{p}K = \mathrm{Q}(A/\mathfrak{p})$. Furthermore, since $S \subseteq A \smallsetminus \mathfrak{p}$, $\mathfrak{p} = \iota_S^{-1}(\mathfrak{p}K)$ and $\iota_S^{-1}(K \smallsetminus \mathfrak{p}K) = A \smallsetminus \mathfrak{p}$. Therefore $K_{\mathfrak{p}K} = A_{\mathfrak{p}} = \mathrm{Q}(A/\mathfrak{p})$ (the second equality by assumption in (ii)) is an integral domain. Now, (iii) follows immediate from the following Exercise:

Exercise: Let A be a commutative ring and let $\mathcal{M}(A) = \mathrm{Min}\,(\mathrm{Spec}\,A, \subseteq)$ be the set of all minimal prime ideals in A. Then:

(1) If $A_{\mathfrak{p}}$ is an integral domain for every $\mathfrak{p} \in \mathrm{Spec}\,A$, then $\mathfrak{p} \in \mathcal{M}(A)$ are pairwise comaximal.

(2) The following statements are equivalent:

(i) $A_{\mathfrak{p}}$ is an integral domain for every $\mathfrak{p} \in \mathrm{Spec}\,A$ and $\mathcal{M}(A)$ is finite.

(ii) $A = A_1 \times \cdots \times A_n$ is a finite product of integral domains $A_1, \ldots, A_n$. Moreover, in this case, $A_i = A/\mathfrak{p}_i$ with $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} = \mathcal{M}(A)$.

Proof: (1) Suppose that $\mathfrak{p}, \mathfrak{q} \in \mathcal{M}(A)$ are *not* comaximal, i. e. $\mathfrak{p} + \mathfrak{q} \subseteq \mathfrak{m}$ for some $\mathfrak{m} \in \operatorname{Spm} A$. Then $A_\mathfrak{m}$ contains two minimal prime ideals $\mathfrak{p}A_\mathfrak{m}$ and $\mathfrak{q}A_\mathfrak{m}$. But $A_\mathfrak{m}$ is an integral domain by assumption and so (0) is its only minimal prime ideal. Therefore $\mathfrak{p}A_\mathfrak{m} = \mathfrak{q}A_\mathfrak{m}$ and hence $\mathfrak{p} = \mathfrak{q}$.

(2) (i)⇒(ii): Note that, since $(\mathfrak{n}_A)_\mathfrak{m} = \mathfrak{n}_{A_\mathfrak{m}}) = 0$ (since $A_\mathfrak{m}$ is an integral domain by assumption (i)) for all $\mathfrak{m} \in \operatorname{Spm} A$, by local global principle, (see Exercise 7.26) $\mathfrak{n}_A = 0$, i. e. A is reduced. Now, since $\mathcal{M}(A)$ is finite, the canonical homomorphism $A \longrightarrow \prod_{\mathfrak{p} \in \mathcal{M}(A)} A/\mathfrak{p}$, $a \mapsto (\overline{a}_\mathfrak{p})_{\mathfrak{p} \in \mathcal{M}(A)}$, is injective $(\operatorname{Ker} \varphi = \cap_{\mathfrak{p} \in \mathcal{M}(A)} \mathfrak{p} = \mathfrak{n}_A = 0$, since A is reduced) and, by (1) and the Chinese Remainder Theorem, is surjective. Therefore A is product of integral domains.

(ii)⇒(i): Assume that $A = \prod_{i=1}^n A_i$ with A_i integral domain for all $i = 1, \ldots, n$. Let $\mathfrak{p} \in \operatorname{Spec} A$. Then $A_\mathfrak{p} = \prod_{i=1}^n (A_i)_\mathfrak{p}$ and hence $A_\mathfrak{p} = (A_i)_\mathfrak{p}$ for some $1 \le i \le n$, since $A_\mathfrak{p}$ is local and $A_i)\mathfrak{p}$ are integral domains for all $i = 1, \ldots, n$. This proves that $A_\mathfrak{p}$ is an integral domain. Further, note that each $\mathfrak{p} \in \mathcal{M}(A)$ is of the form (see for example, Exercise 8.20 below) $\mathfrak{p} = \prod_{i=1}^n \mathfrak{a}_i$ with $\mathfrak{a}_i = 0$ for some (unique) $1 \le i \le n$ and $\mathfrak{a}_j = A_j$ for all $j = 1, \ldots, n$, $j \neq i$. Therefore the i-th projection $A = \prod_{i=1}^n A_i \to A_i$ induces an isomorphism $A/\mathfrak{p}_i \xrightarrow{\sim} A_i$. This proves (ii).)

8.12 Let A be a ring.

(a) For an element $f \in A$, show the following two conditions are equivalent:
(i) $\operatorname{D}(f)$ is dense in $\operatorname{Spec} A$. (ii) The residue class of f is a non-zero divisor in $A_{\rm red} = A/\mathfrak{n}_A$.
— In particular: If f is a non-zero divisor in A, then $\operatorname{D}(f)$ is dense in $\operatorname{Spec} A$. Give an example which shows that the converse is not true in general.

(**Proof:** Let $\mathcal{M}(A) := \operatorname{Min}(\operatorname{Spec} A, \subseteq)$. Then $\mathfrak{n}_A = \cap_{\mathfrak{p} \in \mathcal{M}(A)} \mathfrak{p}$. Further, $A_{\rm red} = A/\mathfrak{n}_A$ is reduced and the image of $\cup_{\mathfrak{p} \in \mathcal{M}(A)} \mathfrak{p}$ in $A_{\rm red}$ is the set $\operatorname{ZDiv} A_{\rm red}$ of zerodivisors in $A_{\rm red}$, see Exercise 8.10.

(i)⇒(ii): Suppose that $g \in A$ with $fg = 0$ in $A_{\rm red}$, i. e. $fg \in \mathfrak{n}_A = \cap_{\mathfrak{p} \in \mathcal{M}(A)} \mathfrak{p}$. Then $\emptyset = \operatorname{D}(fg) = \operatorname{D}(f) \cap \operatorname{D}(g)$ and hence $\operatorname{D}(g) = \emptyset$, since $\operatorname{D}(f)$ is dense in $\operatorname{Spec} A$. Therefore $g \in \mathfrak{p}$ for every $\mathfrak{p} \in \operatorname{Spec} A$. In particular, $g \in \mathfrak{n}_A$, i. e. $g = 0$ in $A_{\rm red}$.

(ii)⇒(i): Suppose that $f \in \operatorname{ZDiv} A_{\rm red} = \cup_{\mathfrak{p} \in \mathcal{M}(A)} \mathfrak{p}$ and that $\operatorname{D}(f)$ is *not* dense in $\operatorname{Spec} A$. Then $\emptyset = \operatorname{D}(f) \cap \operatorname{D}(g) = \operatorname{D}(fg)$, i. e. $fg \in \mathfrak{p}$ for every $\mathfrak{p} \in \operatorname{Spec} A$ for some $g \in A$ with $\operatorname{D}(g) \neq \emptyset$. Therefore, since $f \notin \mathfrak{p}$ for every $\mathfrak{p} \in \mathcal{M}(A)$, it follows that $g \in \mathfrak{p}$ for every $\mathfrak{p} \in \mathfrak{n}_A$. In particular, $g \in \mathfrak{p}$ for every $\mathfrak{p} \in \operatorname{Spec} A$ and hence $\operatorname{D}(g) = \emptyset$, a contradiction.

Let $A := K[X,Y]/(X) \cap (X,Y)^2 = K[x,y]$. Then y is a zerodivisor in A, since $xy = 0$, $x \neq 0$, $y \neq 0$, and $\operatorname{D}(y)$ is dense in $\operatorname{Spec} A = \operatorname{Spec} A_{\rm red} = \operatorname{Spec} K[y] = \operatorname{Spec} K[Y]$.
— **Remark:** Elements in A fulfilling conditions (i) and/or (ii) above are called a c t i v e. Non-zerodivisors are active.)

(b) If A is Noetherian and if the open set $U \subseteq \operatorname{Spec} A$ is dense in $\operatorname{Spec} A$, then there exists $f \in A$ such that $\operatorname{D}(f) \subseteq U$ and $\operatorname{D}(f)$ is dense in $\operatorname{Spec} R$.

(**Proof:** Suppose that $U = \operatorname{Spec} A \smallsetminus \operatorname{V}(\mathfrak{a})$ is dense in $\operatorname{Spec} A$. Then $\mathfrak{a} \neq 0$. Further, *we claim that* $\mathfrak{a} \not\subseteq \mathfrak{p}_i$ *for every* $i = 1, \ldots, r$, *where* $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\} = \operatorname{Min}(\operatorname{Spec} A, \subseteq)$ (which is a finite set, since A is noetherian, see Exercise 8.11 (a)). For, if $\mathfrak{a} \subseteq \mathfrak{p}_i$ for some $1 \le i \le r$. Then $U = \operatorname{Spec} A \smallsetminus \operatorname{V}(\mathfrak{a}) \subseteq Y := \cup_{j=1,\, j \neq i}^r \operatorname{V}(\mathfrak{p}_j) \neq \operatorname{Spec} A$ and Y is a closed subset in $\operatorname{Spec} A$ which contradicts the assumption that U is dense in $\operatorname{Spec} A$. This proves the claim $\mathfrak{a} \not\subseteq \mathfrak{p}_i$ for every $i = 1, \ldots, r$. Now, by Prime avoidance Lemma (Exrecise 1.9 (b)) choose $f \in \mathfrak{a} \smallsetminus \left(\cup_{i=1}^r \mathfrak{p}_i\right)$. Then $\operatorname{D}(f) \subseteq U$, since $f \in \mathfrak{a}$, and further, f is a non-zerodivisor in $A_{\rm red}$, since $f \notin \cup_{i=1}^r \mathfrak{p}_i$ and hence by part (a) $\operatorname{D}(f)$ is dense in $\operatorname{Spec} A$.)

8.13 Let A be a ring and $X := \operatorname{Spec} A$. For closed sets $Y_1, Y_2 \subseteq X$, show that the following statements are equivalent:

(i) $Y_1 \uplus Y_2 = X$, i. e. $Y_1 \cup Y_2$ and $Y_1 \cap Y_2 = \emptyset$.
(ii) There are complementary idempotents $e_1, e_2 \in A$ (i. e. idempotents with $e_1 + e_2 = 1$ and $e_1 e_2 = 0$) with $\mathrm{V}(\langle e_i \rangle) = Y_i$, $i = 1, 2$.
(iii) There are comaximal ideals $\mathfrak{a}_1$, $\mathfrak{a}_2 \subseteq A$ with $\mathfrak{a}_1 \mathfrak{a}_2 = 0$ and $\mathrm{V}(\langle e_i \rangle) = Y_i$, $i = 1, 2$.
(iv) There are ideals $\mathfrak{a}_1$, $\mathfrak{a}_2 \subseteq A$ with $\mathfrak{a}_1 \oplus \mathfrak{a}_2 = A$ and $\mathrm{V}(\mathfrak{a}_i) = Y_i$, $i = 1, 2$.
Moreover, given any e_i and $\mathfrak{a}_i$, $i = 1, 2$, satisfying (ii) and either (iii) or (iv), necessarily $e_i \in \mathfrak{a}_i$, $i = 1, 2$.

8.14 Let V be an A-module and let $\operatorname{Supp} V := \{\mathfrak{p} \in \operatorname{Spec} A \mid V_{\mathfrak{p}} \neq 0\}$.

(a) $V \neq 0$ if and only if $\operatorname{Supp} V \neq \emptyset$.

(b) If $\mathfrak{a}$ is an ideal in A, then $\mathrm{V}(\mathfrak{a}) = \operatorname{Supp} A/\mathfrak{a}$.

(c) If $\mathfrak{p} \in \operatorname{Supp} V$, then $\mathrm{V}(\mathfrak{p}) \subseteq \operatorname{Supp} V$.

(d) If $\operatorname{Ann}_A V \cap (A \smallsetminus \mathfrak{p}) \neq \emptyset$, then $\mathfrak{p} \notin \operatorname{Supp} V$. The converse holds if V is a finite A-module.

(e) $\operatorname{Supp} V \subseteq \mathrm{V}(\operatorname{Ann}_A V)$ and the equality holds if V is a finite A-module. In particular, if V is a finite A-module then $\operatorname{Supp} V$ is a closed subset in the Zariski topology on $\operatorname{Spec} A$.

8.15 Let V be an A-module. Prove that :

(a) If $0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$ is a short exact sequence of A-modules, then
$$\operatorname{Supp}, V = \operatorname{Supp} V' \cup \operatorname{Supp} V''.$$

(b) If $V = \sum_{i \in I} V_i$ the sum of the family V_i, $i \in I$ A-submodules of V, then
$$\operatorname{Supp} V = \bigcup_{i \in I} \operatorname{Supp} V_i.$$

(c) If V is finite A-module and if $\mathfrak{a}$ is an ideal in A, then $\operatorname{Supp}(V/\mathfrak{a}V) = \mathrm{V}(\mathfrak{a} + \operatorname{Ann}_A V)$.

(d) Find the support of the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$. Is it closed in the Zariski topology on $\operatorname{Spec} \mathbb{Z}$? (**Hint :** $\operatorname{Supp} \mathbb{Q}/\mathbb{Z} = \mathbb{P}$ (the set of all prime numbers), since $\mathbb{Z}_{(p)} \neq \mathbb{Q}_{(p)} = \mathbb{Q}$ and $\mathbb{Z}_{(0)} = \mathbb{Q}$. In particular, $\operatorname{Supp} \mathbb{Q}/\mathbb{Z}$ is not closed and hence $\operatorname{Supp} \mathbb{Q}/\mathbb{Z} \neq \mathrm{V}(\operatorname{Ann}_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z})$.)

8.16 Let $\varphi : A \to A'$ be a ring homomorphism and let $\varphi^* : \operatorname{Spec} A' \longrightarrow \operatorname{Spec} A$ be the map associated to φ on spectra. Prove that :

(a) Every $\mathfrak{p} \in \operatorname{Spec} A$ is a contraction of some $\mathfrak{p}' \in \operatorname{Spec} A'$ if and only if φ^* is surjective.

(b) If every $\mathfrak{p}' \in \operatorname{Spec} A'$ is an extension of some $\mathfrak{p} \in \operatorname{Spec} A$, then φ^* is injective. Is the converse true?

8.17 (Locally finitely generated and presented modules) Let A be a ring. An A-module V is called locally finitely generated if each $\mathfrak{p} \in \operatorname{Spec} A$ has a neighborhood on which V becomes finitely generated ; more precisely, there exists $f \in A \smallsetminus \mathfrak{p}$ such that V_f is finitely generated over A_f . It is enough that such an f exist for each maximal ideal $\mathfrak{m} \in \operatorname{Spm} A$, since every prime ideal $\mathfrak{p}$ is contained in some maximal ideal $\mathfrak{m}$. Similarly, we define the properties locally finitely presented, locally free of finite rank, and locally free of rank n.

(a) If V is a locally finitely generated A-module, then V is finitely generated. (**Hint :** Note that a family $x_i \in V$, $i \in I$, generated V if and only if for every maximal ideal $\mathfrak{m} \in \operatorname{Spm} A$, the images $x_i/1 \in V_{\mathfrak{m}}$, $i \in I$, generated $V_{\mathfrak{m}}$. Use the fact that $X = \operatorname{Spec} A$ is quasi-compact.)

(b) If V is a locally finitely presented A-module, then V is finitely presented.

(c) For an A-module P the following statements are equivalent:

(i) P is finitely generated and projective.
(ii) P is finitely presented and $P_{\mathfrak{m}}$ is free over $A_{\mathfrak{m}}$ for every $\mathfrak{m} \in \operatorname{Spm} A$.
(iii) P is locally free of finite rank.
(iv) P is finitely presented and for each $\mathfrak{p} \in \operatorname{Spec} A$, there are $f \in A$ and $n \in \mathbb{N}$ such that $\mathfrak{p} \in \mathrm{D}(f)$ and $P_{\mathfrak{q}}$ is free of rank n over $A_{\mathfrak{q}}$ at each $\mathfrak{q} \in \mathrm{D}(f)$. (**Hint:** Using the parts (a), (b) above, Exercise 7.22 and the following Exercise, prove the implications: (i) $\Longleftrightarrow$ (ii)$\Rightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(ii).

Exercise: Let V be a finite A-module and let $S \subseteq A$ be a multiplicatively closed subset.
(a) Let $x_1, \ldots, x_n \in V$. If the images $x_1/1, \ldots, x_n/1 \in S^{-1}V$ generate $S^{-1}V$ over $S^{-1}A$, then there exists $f \in S$ such that $x_1/1, \ldots, x_n/1 \in V_f$ generate V_f over A_f.
(b) If V is finitely presented and if $S^{-1}V$ is a free $S^{-1}A$-module of rank n, then there exists $f \in S$ such that V_f is afree A_f-module of rank n.)

8.18 Let A be a ring. Show that every non-empty closed subset $V \subseteq \operatorname{Spec} A$ contains a closed point. Deduce that an open subset $U \subseteq \operatorname{Spec} A$ containing all closed points of $\operatorname{Spec} A$ must coincide with $\operatorname{Spec} A$.

8.19 Let K be an *algebraically closed* field, $K[X_1, \ldots, X_n]$ the polynomial ring in n indeterminates $X_1, \ldots, X_n$ over K and let $X := \operatorname{Spec} K[X_1, \ldots, X_n]$. Show that:

(a) The set of closed points in X can canonically be identified with with K^n.

(b) If $n = 1$ then there is exactly one non-closed point in X, namely the generic point of X.

(c) If $n = 2$, then the non-closed points in X that are different from the generic point are given by the principal (prime) ideals $\langle f \rangle$ where $f \in K[X_1, X_2]$ is irreducible and the closure $\overline{\{y\}}$ of such points consists of y as the generic point and of the *curve* $\{x \in K^2 \mid f(x) = 0\}$.

8.20 Let $A_1, \ldots, A_n$ be rings. Show that there is a canonical bijection

$$\operatorname{Spec} \prod_{i=1}^n A_i \xrightarrow{\ \sim\ } \coprod_{i=1}^n \operatorname{Spec} A_i.$$

(**Hint:** Note that the map $\mathfrak{I}(A_1) \times \cdots \times \mathfrak{I}(A_n) \xrightarrow{\sim} \mathfrak{I}(A_1 \times \cdots \times A_n)$, $(\mathfrak{a}_1, \ldots, \mathfrak{a}_n) \longmapsto \mathfrak{a}_1 \times \cdots \times \mathfrak{a}_n$, is bijective. Futher, its inverse induces a canonical bijections: $\operatorname{Spec}(A_1 \times \cdots \times A_n) \xrightarrow{\sim} \coprod_{i=1}^n \operatorname{Spec} A_i$ and $\operatorname{Spm}(A_1 \times \cdots \times A_n) \xrightarrow{\sim} \coprod_{i=1}^n \operatorname{Spm} A_i$.)

8.21 Let A be a finite type algebra over a field K.

(a) Let $Y \subseteq \operatorname{Spec} A$ be a closed subset. Show that the closed points are dense in Y.

(b) Show that $\operatorname{Spec} A$ is finite if and only if A is a finite K-algebra, i. e. $\operatorname{Dim}_K A$ is finite. (**Hint:** Note that A is noetherian and hence A contains only finitely many minimal prime ideals. Use this to reduce the assertion to the case where A is an integral domain.)

8.22 (Functorial properties of Spectra) Every ring homomorphism $\varphi : A \to A'$ induces a map $\varphi^* := \operatorname{Spec} \varphi : X' := \operatorname{Spec} A' \to \operatorname{Spec} A =: X$, $\mathfrak{p}_{x'} \mapsto \varphi^{-1}(\mathfrak{p}_{x'})$ between the associated spectra. More precisely, for every $x' \in X'$, the following diagram is commutative:

$$\begin{array}{ccccc}
A & \xrightarrow{\pi_{\varphi^{-1}(\mathfrak{p}_{x'})}} & A/\varphi^{-1}(\mathfrak{p}_{x'}) & \xhookrightarrow{\ \iota\ } & \kappa(\varphi^*(x')) \\
\Big\downarrow{\scriptstyle \varphi} & & \Big\downarrow{\scriptstyle \varphi_x} & & \Big\downarrow{\scriptstyle \varphi_x} \\
A' & \xrightarrow{\pi_{\mathfrak{p}_{x'}}} & A'/\mathfrak{p}_{x'} & \xhookrightarrow{\ \iota\ } & \kappa(x')
\end{array}$$

In particular, for $f \in A$ and $x' \in X'$, we have :

$$\varphi_x\,(f(\varphi^*(x'))) = \varphi(f)(x')\,, \quad \text{i.e.} \quad f \circ \varphi^* = \varphi(f)\,.$$

Therefore φ might be interpreted as the map composing functions $f \in A$ with φ^*.

Further, $\mathrm{id}_A^* = \mathrm{id}_{\mathrm{Spec}\,A}$ and if $\varphi' : A' \to A''$ is another ring homomorphism, then $(\varphi' \circ \varphi)^* = \varphi^* \circ \varphi'^*$, i. e. the diagram

$$\begin{array}{ccccc} & & \mathrm{Spec}\,A' & & \\ & {\scriptstyle \varphi'^*}\swarrow & & \searrow{\scriptstyle \varphi^*} & \\ \mathrm{Spec}\,A'' & & \xrightarrow{\quad (\varphi'\circ\varphi)^* \quad} & & \mathrm{Spec}\,A \end{array}$$

is commutative.

(a) Let $\varphi : A \longrightarrow A'$ be a ring homomorphism and let $\varphi^* : \mathrm{Spec}\,A' \longrightarrow \mathrm{Spec}\,A$ be the map associated to φ on spectra. Then :

(a.1) $(\varphi^*)^{-1}(\mathrm{V}(E)) = \mathrm{V}(\varphi(E))$ for every subset $E \subseteq A$. In particular, if $\mathfrak{a}$ is an ideal in A, then $(\varphi^*)^{-1}(\mathrm{V}(\mathfrak{a}) = \mathrm{V}(\varphi(\mathfrak{a})A')$.

(a.2) $\overline{\varphi^*\,(\mathrm{V}(\mathfrak{a}'))} = \mathrm{V}\left(\varphi^{-1}(\mathfrak{a}')\right)$ for every ideal $\mathfrak{a}' \subseteq A'$.

(a.3) $(\varphi^*)^{-1}(\mathrm{D}(f)) = \mathrm{D}(\varphi(f))$ for every subset $f \in A$.

(b) The map $\varphi^* : \mathrm{Spec}\,A' \longrightarrow \mathrm{Spec}\,A$ associated to a ring homomorphism $\varphi : A \to A'$ is continuous with respect to the Zariski topologies on $\mathrm{Spec}\,A$ and $\mathrm{Spec}\,A'$.

(c) The assignments $A \vdash\!\!\rightsquigarrow \mathrm{Spec}\,A$, $\varphi \vdash\!\!\rightsquigarrow \varphi^*$ define a contravariant functor from the category $\mathcal{R}ings$ of rings to the category $\mathcal{T}ops$ of topological spaces.

(d) Let A and A' be algebras of finite type over a field K and let $\varphi : A \to A'$ be a K-algebra homomorphism. Then the map $\varphi^* : \mathrm{Spec}\,A' \longrightarrow \mathrm{Spec}\,A$ associated to φ maps the maximal spectrum $\mathrm{Spm}\,A'$ into $\mathrm{Spm}\,A$, i. e. $\varphi^*(\mathrm{Spm}\,A') \subseteq \mathrm{Spm}\,A$ and maps the K-spectrum K-$\mathrm{Spec}\,A'$ into K-$\mathrm{Spec}\,A$, i. e. $\varphi^*(K\text{-}\mathrm{Spec}\,A') \subseteq K$-$\mathrm{Spec}\,A$. Moreover, the assignments $A \vdash\!\!\rightsquigarrow \mathrm{Spm}\,A$, $\varphi \vdash\!\!\rightsquigarrow \varphi^*$ and the assignments $A \vdash\!\!\rightsquigarrow K$-$\mathrm{Spec}\,A$, $\varphi \vdash\!\!\rightsquigarrow \varphi^*$ define a contravariant functors from the category $\mathcal{K}algs$ of K-algebras to the category $\mathcal{T}ops$ of topological spaces (Spm A and K-Spec A are equipped with the induced Zariski topology from Spec$A$, see also R 8.1.3 (e).).

8.23 Let $\varphi : A \longrightarrow A'$ be a ring homomorphism such that every element $f' \in A'$ is of type $f' = \varphi(f)\,u$ with $f \in A$ and $u \in (A')^\times$. Then the $\varphi^* : \mathrm{Spec}\,A' \longrightarrow \mathrm{Spec}\,A$ associated to φ is injective and induces a homeomorphism $\mathrm{Spec}\,A' \xrightarrow{\sim} \mathrm{Img}\,\varphi^* \subseteq \mathrm{Spec}\,A$ where $\mathrm{Spec}\,A$ and $\mathrm{Spec}\,A'$ are equipped with their Zariski topologies and $\mathrm{Img}\,\varphi^*$ with the subspace topology induced from the Zariski topology on $\mathrm{Spec}\,A$.

There are two typical examples of ring homomorphisms $\varphi : A \to A'$ where the assumption in the above assertion is fulfilled, namely, residue-class homomorphisms and localizations.

(a) Let A be a ring and let $\mathfrak{a} \subseteq A$ be an ideal. Then the map $\pi_{\mathfrak{a}}^* : \mathrm{Spec}\,A/\mathfrak{a} \longrightarrow \mathrm{Spec}\,A$ associated to the residue-class homomorphism $\pi_{\mathfrak{a}} : A \longrightarrow A/\mathfrak{a}$ induces a homeomorphism

$$\mathrm{Spec}\,A/\mathfrak{a} \xrightarrow{\sim} \mathrm{V}(\mathfrak{a}) \subseteq \mathrm{Spec}\,A.$$

(In this case $\pi_{\mathfrak{a}}^*$ is called a c l o s e d i m m e r s i o n of spectra.)

(b) Let A be a ring and let $S \subseteq A$ be a multiplicatively closed subset in A. Then the map $\iota_S^* : \operatorname{Spec} S^{-1}A \longrightarrow \operatorname{Spec} A$ associated to the canonical homomorphism $\iota_S : A \longrightarrow S^{-1}A$ induces a homeomorphism

$$\operatorname{Spec} S^{-1}A \xrightarrow{\ \sim\ } \bigcap_{f\in S} \mathrm{D}(f) \subseteq \operatorname{Spec} A.$$

In particular, $\mathfrak{p} \in \operatorname{Spec} A$, $S_{\mathfrak{p}} := A \smallsetminus \mathfrak{p}$, $\operatorname{Img} \iota_{S_{\mathfrak{p}}}^* = \bigcap_{f\in S_{\mathfrak{p}}} \mathrm{D}(f)$. (If Img ι_S^* is open in $\operatorname{Spec} A$, then ι_S^* is called an open immersion of spectra. For example, the latter is the case if S is generated by finitely many elements $f_1,\ldots,f_r \in A$, since then $\bigcap_{f\in S} \mathrm{D}(f) = \mathrm{D}(f_1\cdot\ldots\cdot f_r)$ is open in $\operatorname{Spec} A$.)

8.24 Let X,Y,T be indeterminates over $\mathbb{C}$, $A := \mathbb{C}[X,Y]$, $B := \mathbb{C}[X,Y,T]/\langle XT-Y\rangle = \mathbb{C}[x,y]$ and let $\pi := \iota^* : \mathbb{C}\text{-Spec } B \to \mathbb{C}\text{-Spec } A = \mathbb{C}^2$ be the map associated to the inclusion $\iota : A \hookrightarrow B$ on the $\mathbb{C}$-spectra. Show that π induces a homeomorphism $\pi^{-1}(\mathrm{D}(X)) \xrightarrow{\sim} \mathrm{D}(X)$ and hence an open immersion $\pi^{-1}(\mathrm{D}(X) \to \mathbb{C}^2$. Further, prove that the fibre $\pi^{-1}((0,0))$ is homeomorphic to $\mathbb{C}$ and check that $\operatorname{Img}\pi = \mathrm{D}(X) \cup \{(0,0)\}$.

8.25 Let $\varphi : A \to B$ be a ring homomorphism and and let $\varphi^* : \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ be the map associated to φ on spectra. Then:

(a) If φ is surjective, then φ^* is a homeomorphism of $\operatorname{Spec} B$ onto the closed subset $\mathrm{V}(\operatorname{Ker}\varphi)$ of $\operatorname{Spec} A$. In particular, $\operatorname{Spec} A$ and $\operatorname{Spec} A/\mathfrak{n}_A$ (where $\mathfrak{n}_A$ is the nil-radical of A) are canonically homeomorphic. See also Exercise 8.8 (b).

(b) (Dominant morphisms) We say that φ^* is dominant if the image $\varphi^*(\operatorname{Spec} B)$ is dense in $\operatorname{Spec} A$, i. e. $\overline{\varphi^*(\operatorname{Spec} B)} = \operatorname{Spec} A$. Equivalently, the kernel $\operatorname{Ker}\varphi \subseteq \operatorname{nil}(A)$. In particular, if φ is injective, then φ^* is dominant.

8.26 Let K be a field. Let $V \subseteq K^n$ and $W \subseteq K^m$ be two affine K-algebraic subsets with K-coordinate rings $K[V] := K[X_1,\ldots,X_n]/\mathrm{I}_K(V)$ and $K[W] := K[X_1,\ldots,X_m]/\mathrm{I}_K(V)$, respectively. Furthermore, let $\varphi : K[W] \to K[V]$ be a K-algebra homomorphism and let $\varphi^* : V \cong K\text{-Spec } K[V] \to K\text{-Spec } K[W] \cong W$ be the morphism associated to φ. Prove that:

(a) φ^* is dominant (see Exercise 8.25 (b)), i. e. the image $\varphi^*(V)$ is dense in W if and only if φ is injective. Give an example in which φ^* is dominant but not surjective.

(b) If φ is surjective, then φ^* is injective. Is the converse true?

(**Hint:** Let $\varphi : K[X] \xrightarrow{\iota} K[X]_X \xrightarrow{\sim} K[X,Y]/\langle XY-1\rangle$. Then φ is injective but not surjective. Further, the associated map $\varphi^* : K\text{-Spec } \mathbb{K}[X,Y]/\langle XY-1\rangle \to K\text{-Spec } K[X]$ is dominant but not surjective.)

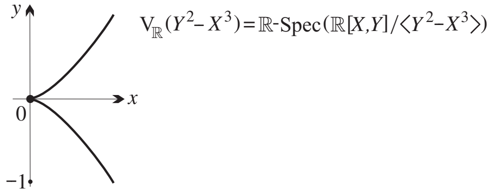
8.27 Let A be an integral domain with $\operatorname{Spec} A = \{0,\mathfrak{p}\}$ (for example, a formal power series ring $A = K[\![X]\!]$ in one indeterminate X over a field) and the quotient field K. Further, let $B := A/\mathfrak{p} \times K$ be the product ring and let $\varphi : A \to B$ be the ring homomorphism defined by $\varphi(x) = (\pi(x),x)$ where $\pi(x)$ is the residue-class of x modulo $\mathfrak{p}$. Show that the associated map $\varphi^* : \operatorname{Spec} B \to \operatorname{Spec} A$ on the spectra is bijective, but not a homeomorphism.

8.28 Let $\varphi : A \longrightarrow A'$ be a ring homomorphism and let $\varphi^* : \operatorname{Spec} A' \longrightarrow \operatorname{Spec} A$ be the map associated to φ on spectra. Suppose that $X' = \operatorname{Spec} A'$ is irreducible and let $x' \in X'$ be its generic point. Show that $\varphi^*(x')$ is the generic point of the closure $\overline{\operatorname{Img}\varphi^*}$ of $\operatorname{Img}\varphi^*$. See Exercise 8.9 (c).

8.29 Let K be a field, $\varphi : A \longrightarrow A'$ be a K-algebra homomorphism of K-algebras and let $\varphi^* : \operatorname{Spec} A' \longrightarrow \operatorname{Spec} A$ be the map associated to φ on spectra. If A' is finite type over

K, then image of every closed point $x' \in \operatorname{Spec} A'$ is again a closed point in $\operatorname{Spec} A$. Is the assumption that A' is finite type over K necessary?

8.30 (Neil's Parabola) Let K be a field, X, Y, T be indeterminates over K and let $\varphi : K[x,y] := K[X,Y]/\langle Y^2 - X^3 \rangle \longrightarrow K[T]$ be the K-algebra homomorphism defined by $x \mapsto T^2$, $y \mapsto T^3$. Show that the associated map $\varphi^* : \operatorname{Spec} A' \to \operatorname{Spec} A$ on the spectra is a homeomorphism, although φ is injective, but not surjective.



Neil's Parabola ---Parabola cuspidata

(**Hint:** The K-algebra homomorphism $S_x^{-1}\varphi : S_x^{-1}(K[x,y]) \longrightarrow S_x^{-1}(K[T])$, where $S_x := \{x^n \mid n \in \mathbb{N}\}$, is an isomorphism. — The plane curve with equation $y^2 = x^3$ has been considered by William Neil[7] in 1657 and is called the *semicubical* or *Neil's parabola*.)

8.31 (Fibres of a homomorphism) Let $\varphi : A \to B$ be a ring homomorphism and let $\varphi^* : \operatorname{Spec} B \to \operatorname{Spec} A$ be the map associated to φ on spectra. For $\mathfrak{p} \in \operatorname{Spec} A$, the set $(\varphi^*)^{-1}(\mathfrak{p}) := \{\mathfrak{q} \in \operatorname{Spec} B \mid \varphi^{-1}(\mathfrak{q}) = \mathfrak{p}\}$ is called the fibre of φ^* over $\mathfrak{p}$.

For example, if $\varphi = \iota_S : A \to S^{-1}A$ is a localization homomorphism, then the fibre over $\mathfrak{p} \in \operatorname{Spec} A$ is $\{S^{-1}\mathfrak{p}\}$ if $S \cap \mathfrak{p} = \emptyset$, and $\emptyset$ if $S \cap \mathfrak{p} \neq \emptyset$. If $\varphi = \pi_{\mathfrak{a}} A \to A/\mathfrak{a}$ is a residue-class homomorphism, then fibre over $\mathfrak{p}\operatorname{Spec} A$ is $\{\mathfrak{p}/\mathfrak{a}\}$ if $\mathfrak{a} \subseteq \mathfrak{p}$, and $\emptyset$ if $\mathfrak{a} \not\subseteq \mathfrak{p}$.

(a) Let $\mathfrak{p} \in \operatorname{Spec} A$ be such that $\mathfrak{p}B_{\mathfrak{p}} \neq B_{\mathfrak{p}}$. Then the map

$$\mathrm{V}(\mathfrak{p}B_{\mathfrak{p}}) \xrightarrow{(\iota_{\mathfrak{p}}^B)^*} (\varphi^*)^{-1}(\mathfrak{p}), \quad \mathfrak{q}' \longmapsto (\iota_{\mathfrak{p}}^B)^{-1}(\mathfrak{q}'),$$

is bijective. (**Hint:** For $\mathfrak{p} \in \operatorname{Spec} A$, let $\iota_{\mathfrak{p}}^A : A \to A_{\mathfrak{p}}$ and $\iota_{\mathfrak{p}}^B : B \to B_{\mathfrak{p}}$ be the natural localization homomorphisms. From the following commutative diagrams

$$\begin{array}{ccc} A & \xrightarrow{\iota_{\mathfrak{p}}^A} & A_{\mathfrak{p}} \\ \varphi \downarrow & & \downarrow \varphi_{\mathfrak{p}} \\ B & \xrightarrow{\iota_{\mathfrak{p}}^B} & B_{\mathfrak{p}} \end{array} \qquad \begin{array}{ccc} \mathfrak{p} \in \operatorname{Spec} A & \xleftarrow{(\iota_{\mathfrak{p}}^A)^*} & \operatorname{Spec} A_{\mathfrak{p}} \ni \mathfrak{p}A_{\mathfrak{p}} \\ \varphi^* \uparrow & & \uparrow (\varphi_{\mathfrak{p}})^* \\ \mathfrak{q} = (\iota_{\mathfrak{p}}^B)^{-1}(\mathfrak{q}') \in \operatorname{Img}(\iota_{\mathfrak{p}}^B)^* \subseteq \operatorname{Spec} B & \xleftarrow{(\iota_{\mathfrak{p}}^B)^*} & \operatorname{Spec} B_{\mathfrak{p}} \supseteq \mathrm{V}(\mathfrak{p}B_{\mathfrak{p}}) \ni \mathfrak{q}' \end{array}$$

It follows that

$$\mathrm{V}(\mathfrak{p}B_{\mathfrak{p}}) \xrightarrow[\sim]{(\iota_{\mathfrak{p}}^B)^*} \operatorname{Img}(\iota_{\mathfrak{p}}^B)^* = \{(\iota_{\mathfrak{p}}^B)^{-1}(\mathfrak{q}') =: \mathfrak{q} \in \operatorname{Spec} B \mid \mathfrak{p}B \subseteq \mathfrak{q} \text{ and } \mathfrak{q} \cap (A \smallsetminus \mathfrak{p}) = \emptyset)\} \xrightarrow[\sim]{\varphi^*} (\varphi^*)^{-1}(\mathfrak{p})$$

and $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$. Therefore the map $\mathrm{V}(\mathfrak{p}B_{\mathfrak{p}}) \xrightarrow{(\iota_{\mathfrak{p}}^B)^*} (\varphi^*)^{-1}(\mathfrak{p})$, $\mathfrak{q}' \longmapsto (\iota_{\mathfrak{p}}^B)^{-1}(\mathfrak{q}')$ is bijective.)

(b) Let $\mathfrak{p} \in \operatorname{Spec} A$. Then the map $(\varphi^*)^{-1}(\mathfrak{p}) \xrightarrow{\sim} \operatorname{Spec} B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$, $\quad \mathfrak{q} \longmapsto \mathfrak{q}B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$, is a homeomorphism. In particular, if B is finite over A, then the fibres of φ^* are noetherian subspaces of $\operatorname{Spec} B$. (**Hint:** For every $\mathfrak{p} \in \operatorname{Spec} A$, the $\kappa(\mathfrak{p})$-algebra $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$ is finite.)

(c) For $\mathfrak{p} \in \operatorname{Spec} A$, the following statements are equivalent:

(i) The fibre $(\varphi^*)^{-1}(\mathfrak{p}) \neq \emptyset$. (ii) $\varphi^{-1}(\mathfrak{p}B) = \mathfrak{p}$. (iii) $B_{\mathfrak{p}} \neq \mathfrak{p}B_{\mathfrak{p}}$.

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[7] William Neil (1637-1670) was an English mathematician and founder member of the *Royal Society*. — The oldest national institution, formally known as the *Royal Society of London*, founded in 1660, for promoting science and its benefits, recognising excellence in science, supporting outstanding science and education.

(**Hint :** The implications (i)⇒(ii)⇒(iii) are easy to prove and for (iii)⇒(i) use the part (a).)

(d) Suppose that φ is faithfully flat. Then φ^* is surjective. (**Hint :** Verify the condition (ii) in the part (c).)

8.32 Let K be a field, $X_1,\ldots,X_n,Y_1,\ldots,Y_m$ be indeterminates over K, and let $F_1,\ldots,F_m \in K[X_1,\ldots,X_n]$. Further, let $\varepsilon : K[Y_1,\ldots,Y_m] \longrightarrow K[X_1,\ldots,X_n]$ be the substitution K-algebra homomorphism with $\varepsilon(Y_1)=F_1,\ldots,\varepsilon(Y_m)=F_m$, and let

$$\varepsilon^* : K^n = K\text{-Spec}\,K[X_1,\ldots,X_n] \longrightarrow K\text{-Spec}\,K[Y_1,\ldots,Y_m] = K^m$$

be the map associated to ε on the K-spectra. It is the polynomial map on K^n defined by $a \longmapsto (F_1(a),\ldots,F_m(a))$, $a\in K^n$.

The fibre of ε^* over 0 is the (affine) K-algebraic set (in K^n) defined by the $F_1,\ldots,F_m$, i. e.

$$(\varepsilon^*)^{-1}(0) = \mathrm{V}_K(F_1,\ldots,F_m) = \{a\in K^n \mid F_1(a)=\cdots=F_m(a)=0\}.$$

More generally, the fibre of ε^* over $b=(b_1,\ldots,b_m)$ is the affine K-algebraic set

$$(\varepsilon^*)^{-1}(b) = \{a\in K^n \mid F_1(a)=b_1\,,\,\ldots\,,F_m(a)=b_m\}.$$

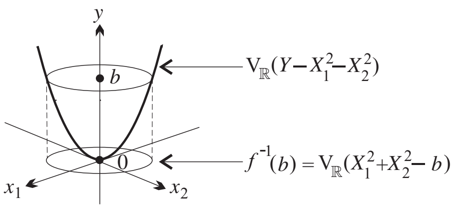
These fibres are described by using the so-called f i b r e a l g e b r a (see the Exercise 8.28).

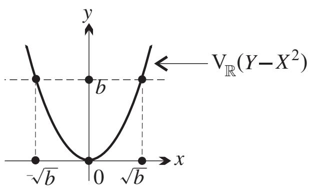
$$K[X_1,\ldots,X_n]/\varepsilon(\mathfrak{m}_b)\,K[X_1,\ldots,X_n] = K[X_1,\ldots,X_n]/\langle F_1-b_1,\ldots,F_m-b_m\rangle$$

of the map ε at the point $b\in K^m$.

(**Remark :** The study of fibres of a polynomial map $K^n\to K^m$ sems to have motivated the definition of affine K-algebraic sets and their study further.)

(a) For the polynomial map $f:\mathbb{R}^2\to\mathbb{R}$, $(x_1,x_2)\mapsto x_1^2+x_2^2$, and $b\in\mathbb{R}$, the fibre $f^{-1}(b)=\mathrm{V}_{\mathbb{R}}(X_1^2+X_2^2-b)$ is a circle if $b>0$, the origin $(0,0)$ if $b=0$ and empty if $b<0$. Note that the prime ideals $\mathfrak{p}_b := \langle X_1^2+X_2^2-b\rangle)$, $b\in\mathbb{R}$, and $\mathfrak{m} := \langle X_1\,,X_2\rangle \in \operatorname{Spec}\mathbb{R}[X_1,X_2]$ are different, but $\mathrm{V}_{\mathbb{R}}(\mathfrak{p}_0)=\mathrm{V}_{\mathbb{R}}(\mathfrak{m})$ and $\mathrm{V}_{\mathbb{R}}(\mathfrak{p}_b)=\emptyset$ if $b<0$. (**Remark :** In general, $\mathrm{V}_{\mathbb{R}}(F_1,\ldots,F_m)=\mathrm{V}_{\mathbb{R}}(F_1^2+\cdots+F_m^2)$ for arbitrary polynomials $F_1,\ldots,F_m\in\mathbb{R}[X_1,\ldots,X_n]$. Thus, *every affine algebraic set in $\mathbb{R}^n$ is the zero set of a single polynomial.*)





(b) Let $g:\mathbb{R}\to\mathbb{R}$ be the polynomial map $x\mapsto x^2$ which is the restriction of the map f of the example (a) to the line $\mathrm{V}_{\mathbb{R}}(X_2)=\{x_2=0\}$. The fibre $g^{-1}(b)=\mathrm{V}_{\mathbb{R}}(X^2-b)=\{\pm\sqrt{b}\}$ has exactly two points if $b>0$; it has exactly one point $\{0\}$ if $b=0$ and it is empty if $b<0$. For these three cases the corresponding fibre algebras $\mathbb{R}[X]/(X^2-b)$, $b\in\mathbb{R}$, are isomorphic to the product algebra $\mathbb{R}\times\mathbb{R}$, to the algebra $\mathbb{R}[\varepsilon]:=\mathbb{R}[X]/(X^2)$ of *dual numbers* and to the algebra $\mathbb{C}$, respectively.

8.33 Let K be a field, B a K-algebra of finite type and let $\mathfrak{q}\in\operatorname{Spec}B$. The the following statements are equivalent :

(i) $\mathfrak{q}$ is isolated in $\operatorname{Spec}B$, i. e. $\{\mathfrak{q}\}$ is open in $\operatorname{Spec}B$.

(ii) There exists $f\in B$ such that $\mathrm{D}(f)=\{\mathfrak{q}\}$.

(iii) $B_{\mathfrak{q}}$ is a finite K-algebra.

(**Hint :** (ii)⇒(iii) : Note that (ii) implies that $B_{\mathfrak{q}}$ is local Artinian with the maximal ideal $\mathfrak{q}B_f$

and hence $B_f/\mathfrak{q}B_f$ is a finite K-algebra (by HNS 3). Therefore B_f is finite over K. For (iii)⇒(ii) consider the exact sequence $0 \longrightarrow \text{Ker}\ \iota_{\mathfrak{q}} \longrightarrow B \xrightarrow{\iota_{\mathfrak{q}}} B_{\mathfrak{q}} \longrightarrow \text{Coker}\ \iota_{\mathfrak{q}} \longrightarrow 0$ of B-modules. Since $\text{Supp}\,(\text{Ker}\ \iota_{\mathfrak{q}})$ and $\text{Supp}\,(\text{Coker}\,\iota_{\mathfrak{q}})$ are closed subsets of $\text{Spec}\,B$ (see Exercise 8.14 (e)), there exists $f \in B_{\mathfrak{q}}$ such that $(\text{Ker}\ \iota_{\mathfrak{q}})_f = 0$ and $(\text{Coker}\ \iota_{\mathfrak{q}})_f = 0$. This proves the equality $\text{Spec}\,B_f = \{\mathfrak{q}B_f\}$.)

8.34 Let B be an A-algebra of finite type over the ring A with the structure homomorphism $\varphi : A \to B$ and let $\varphi^* : \text{Spec}\,B \longrightarrow \text{Spec}\,A$ be the map associated to φ on spectra.

(a) Show that the following statements are equivalent :

(i) The fibres of φ^* are discrete.

(ii) For every $\mathfrak{p} \in \text{Spec}\,A$ $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$ is finite over the residue field $\kappa(\mathfrak{p})$ at $\mathfrak{p}$.

In particular, the fibres of φ^* are finite.

(b) For a noetherain ring A show that the following statements are equivalent :

(i) A is Artinian.

(ii) $\text{Spec}\,A$ is discrete and finite.

(iii) $\text{Spec}\,A$ is discrete.

8.35 Let B be a *flat* A-algebra with the structure homomorphism $\varphi : A \to B$ and let φ^* : $\text{Spec}\,B \longrightarrow \text{Spec}\,A$ be the map associated to φ on spectra. Then the following statements are equivalent :

(i) $(\mathfrak{a}B) \cap A = \mathfrak{a}$ for every ideal $\mathfrak{a}$ in A.

(ii) The map $\varphi^* : \text{Spec}\,B \to \text{Spec}\,A$ is surjective.

(iii) For every maximal ideal $\mathfrak{m} \in \text{Spm}\,A$, we have $\mathfrak{m}B \neq B$.