E0 219 Linear Algebra and Applications / August-December 2016

(ME, MSc. Ph. D. Programmes)

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Lectures : Monday and Wednesday ; 11:00–12:30   **Venue:** CSA, Lecture Hall (Room No. 117)

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Midterms : **1-st Midterm :** Saturday, September 17, 2016; 15:00 – 17:00   **2-nd Midterm :** Saturday, October 22, 2016; 15:00 – 17:00

Final Examination : December ??, 2016, 09:00 – -12:00

Evaluation Weightage : Assignments : 20%   **Midterms (Two) :** 30%   **Final Examination :** 50%

Range of Marks for Grades (Total 100 Marks)

	Grade S	Grade A	Grade B	Grade C	Grade D	Grade F
Marks-Range	> 90	76 — 90	61 — 75	46 — 60	35 — 45	< 35

	Grade A$^+$	Grade A	Grade B$^+$	Grade B	Grade C	Grade D	Grade F
Marks-Range	> 90	81 — 90	71 — 80	61 — 70	51 — 60	40 — 50	< 40

4. Dimension of vector spaces

Submit a solution of the $*$-Exercise ONLY.   **Due Date :** Wednesday, 31-08-2016 (Before the Class)

Let K be arbitrary field and let $\mathbb{K}$ denote either the field $\mathbb{R}$ or the field $\mathbb{C}$.

4.1 Let $\omega \in \mathbb{R}_+^\times$ be a fixed positive real number. For $a \in \mathbb{R}$ and $\varphi \in \mathbb{R}$, let $f_{a,\varphi} : \mathbb{R} \to \mathbb{R}$ be the function defined by $t \mapsto a\sin(\omega t + \varphi)$ and let $W := \{f_{a,\varphi} \mid a, \varphi \in \mathbb{R}\}$. Then W is a $\mathbb{R}$-subspace of the $\mathbb{R}$-vector space $\mathbb{R}^{\mathbb{R}}$ of all $\mathbb{R}$-valued functions on $\mathbb{R}$.

(a) Find a $\mathbb{R}$-basis of the $\mathbb{R}$-subspace W. What is the dimension $\mathrm{Dim}_{\mathbb{R}} W$? (**Hint :** The functions $t \mapsto \sin \omega t$ and $t \mapsto \cos \omega t = \sin(\omega t + \pi/2)$ form a basis of W. — **Remark:** Elements of W are called *harmonic oscillations* with the *circular frequency* ω.)

(b) Show that every $f \neq 0$ function in W has a unique representation

$$f(t) = a\sin(\omega t + \varphi), \qquad a > 0 \quad \text{and} \quad 0 \le \varphi < 2\pi .$$

(**Remark :** This unique a is called the *amplitude* and φ is called the *phase angle* of f. The zero function has the amplitude 0 and an arbitrary phase angle.)

(c) From the amplitudes and the phase angles of two harmonic oscillations f and g, compute the amplitudes and the phase angles of the functions $f \pm g$.

4.2 Let V be a K-vector space of dimension $n \in \mathbb{N}$.

(a) If $H_1, \ldots, H_r$ are hyper-planes in V, then show that $\mathrm{Dim}_K(H_1 \cap \cdots \cap H_r) \ge n - r$.

(b) If $U \subseteq V$ is a subspace of codimension r, then show that there exist r hyper-planes $H_1, \ldots, H_r$ in V such that $U = H_1 \cap \cdots \cap H_r$.

4.3 Let $x_1 = (a_{11}, \ldots, a_{1n}), \ldots, x_n = (a_{n1}, \ldots, a_{nn})$ be elements of $\mathbb{K}^n$ with

$$|a_{ii}| > \sum_{j=1,\, j \neq i}^{n} |a_{ji}| \quad \text{for all } i = 1, \ldots, n .$$

Show that $x_1, \ldots, x_n$ is a basis of $\mathbb{K}^n$. (**Hint :** It is enough to show the linear independence of $x_1, \ldots, x_n$. For this, suppose that $b_1 x_1 + \cdots + b_n x_n = 0$ with $|b_i| \le 1$ for all i and $b_{i_0} = 1$ for some i_0. This already contradicts the give condition for i_0.)

4.4 Let $x_1, \ldots, x_n \in \mathbb{Z}^n$ be arbitrary vectors with integer components. For every $\lambda \in \mathbb{Q} \setminus \mathbb{Z}$, the vectors $x_1 + \lambda e_1, \ldots, x_n + \lambda e_n$ form a basis of $\mathbb{Q}^n$. (**Hint :** Suppose $a_1(x_1 + \lambda e_1) + \cdots + a_n(x_n + \lambda e_n) = 0$ with $a_1, \ldots, a_n \in \mathbb{Z}$ and $\gcd(a_1, \ldots, a_n) = 1$ and use $\lambda \in \mathbb{Q} \setminus \mathbb{Z}$ to contradict $\gcd(a_1, \ldots, a_n) = 1$.)

***4.5** Let K be a field with at least n elements, $n \in \mathbb{N}^*$ and V be a finite dimensional K-vector space. Let $U_1, \ldots, U_n$ be subspaces of V of equal dimension r and $u_{1i}, \ldots, u_{ir}$ be a basis of U_i for $i = 1, \ldots, n$. Show that there exists $t := \mathrm{Dim}_K V - r$ vectors $w_1, \ldots, w_t \in V$ such that which simultaneously extend the given bases $u_{1i}, \ldots, u_{ir}$ of U_i to a basis $u_{i1}, \ldots, u_{ir}, w_1, \ldots, w_t$ of V for every $i = 1, \ldots, n$. (**Hint :** Use Exercise 2.2.)