E0 219 Linear Algebra and Applications / August-December 2016

(ME, MSc. Ph. D. Programmes)

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Lectures : Monday and Wednesday ; 11:00–12:30 **Venue:** CSA, Lecture Hall (Room No. 117)

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Midterms : 1-st Midterm : Saturday, September 17, 2016; 15:00 – 17:00 **2-nd Midterm :** Saturday, October 22, 2016; 15:00 – 17:00

Final Examination : December ??, 2016, 09:00 – -12:00

Evaluation Weightage : Assignments : 20% **Midterms (Two) :** 30% **Final Examination :** 50%

Range of Marks for Grades (Total 100 Marks)						
	Grade S	**Grade A**	**Grade B**	**Grade C**	**Grade D**	**Grade F**
Marks-Range	>90	76 — 90	61 — 75	46 — 60	35 — 45	<35

	Grade A$^+$	**Grade A**	**Grade B$^+$**	**Grade B**	**Grade C**	**Grade D**	**Grade F**
Marks-Range	>90	81 — 90	71 — 80	61 — 70	51 — 60	40 — 50	<40

Supplement 5

Linear Maps

To understand and appreciate the Supplements which are marked with the symbol † one may possibly require more mathematical maturity than one may have! These are steps towards applications to various other branches of mathematics, especially to analysis, number theory and Affine and Projective Geometry.

Participants may ignore these Supplements — altogether or in the first reading!!

S5.1 Determine whether the following maps are $\mathbb{R}$-linear:

(a) $f:\mathbb{R}^2\to\mathbb{R}^2$ with $f(x_1,x_2):=(x_1^2\,,x_2)$.

(b) $f:\mathbb{R}^2\to\mathbb{R}^2$ mit $f(x_1,x_2):=(x_1+1\,,0)$.

(c) $f:\mathbb{R}^2\to\mathbb{R}^2$ with $f(x_1,x_2):=(x_1+x_2\,,x_1)$.

(d) $f:\mathbb{R}^3\to\mathbb{R}^2$ with $f(x_1,x_2,x_3):=(|x_1-x_2|\,,2x_3)$.

(e) $f:\mathbb{R}^3\to\mathbb{R}^2$ with $f(x_1,x_2,x_3):=(3x_1+2x_2\,,x_1+x_3)$.

S5.2 Determine whether the following maps f on the $\mathbb{K}$-vector space $\mathrm{C}^\infty_{\mathbb{K}}(I)$ of infinitely many times differentiable $\mathbb{K}$-valued functions on the interval $I\subseteq\mathbb{R}$ into itself are $\mathbb{K}$-linear:

(a) $f(x):=a_nx^{(n)}+\cdots+a_1\dot{x}+a_0x+b\ \ (a_n,\ldots,a_0,b\in\mathrm{C}^\infty_{\mathbb{K}}(I)$ fixed).

(b) $f(x):=x^2+\dot{x}^2$.

(c) $f(x):=\left(t\mapsto x(t_0)+\int_{t_0}^{t}x(\tau)a(\tau)d\tau\right)\ \ (t_0\in I$ and $a\in\mathrm{C}^\infty_{\mathbb{K}}(I)$ fixed).

S5.3 **(a)** The complex conjugation $z\mapsto\bar{z}$ of $\mathbb{C}$ into itself is $\mathbb{R}$-linear, but not $\mathbb{C}$-linear.

(b) The maps $z\mapsto\operatorname{Re}z$ and $z\mapsto\operatorname{Im}z$ are $\mathbb{R}$-linear forms on $\mathbb{C}$.

S5.4 For the following linear maps f compute the bases for $\operatorname{Ker}f$ and $\operatorname{Im}f$.

(a) $f:\mathbb{R}^3\to\mathbb{R}^3$ with $f(x_1,x_2,x_3):=(x_1+2x_2+x_3,x_1+3x_2+2x_3,x_1+x_2)$.

(b) $f:\mathbb{R}^4\to\mathbb{R}^3$ with $f(x_1,x_2,x_3,x_4):=(x_1+3x_2-2x_3+x_4,x_1+4x_2-x_3+3x_4,2x_1+3x_2-7x_3-4x_4)$.

(c) $f:\mathbb{R}^3\to\mathbb{R}^4$ with $f(x_1,x_2,x_3):=(x_1+3x_2+3x_3,-2x_1-3x_3,-x_1+x_2-x_3,3x_1-x_2+4x_3)$.

(d) $f:\mathbb{R}^5\to\mathbb{R}^4$ with $f(x_1,x_2,x_3,x_4,x_5):=$

$$(2x_1-x_2-x_3+x_4,-x_1+x_3+x_4+x_5,x_2-x_3-x_4,x_1+x_2-2x_3+x_4+2x_5).$$

S5.5 Let $V:=\mathbb{K}[t]$ be the $\mathbb{K}$-vector space of $\mathbb{K}$-valued polynomial functions on $\mathbb{K}$. Which of the following maps $f:V\to V$ are $\mathbb{K}$-linear ? Find the bases for $\operatorname{Ker}f$ and $\operatorname{Im}f$ for those f which are $\mathbb{K}$-linear.

(a) $f(x) := x^{(n)} =$ (the n-th derivative of x, $n \in \mathbb{N}$.)

(b) $f(x) := x(0) + \ddot{x}$.

(c) $f(x) := (t \mapsto \int_0^t \tau \dot{x}(\tau) d\tau)$.

(d) $f(x) := P(D)x$, where $P(t) \in \mathbb{K}[t]$ is a monic polynomial[1] and D is the differential operator $x \mapsto \dot{x}$. (**Remark :** See also Supplement S3.18.)

S5.6 Let $h : D \to D'$ be an arbitrary map. For every field K, the map $h^* : K^{D'} \to K^D$ defined by $g \mapsto g \circ h$ is K-linear. Describe the functions in $\operatorname{Ker} h^*$ and in $\operatorname{Im} h^*$. Show that h^* is injective (resp. surjective) if and only if h is surjective (resp. injective).

S5.7 A map $f : V \to W$ of $\mathbb{Q}$-vector spaces V and W is already $\mathbb{Q}$-linear if it is additive. The corresponding assertion also holds for vector spaces over the fields $\mathbf{K}_p = \mathbb{Z}/\mathbb{Z}p$, where p is a prime number.

S5.8 For every K–vector space V, the map $f \mapsto f(1)$ is a K-isomorphism of $\operatorname{Hom}_K(K, V)$ onto V.

S5.9 Show that the following linear maps $f_i : \mathbb{R}^3 \to \mathbb{R}^2$, $i = 1, 2, 3$, in $\operatorname{Hom}_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^2)$ are linearly independent: $f_1 : (x_1, x_2, x_3) \mapsto (x_1 + x_2 + x_3, x_1 + x_2)$, $f_2 : (x_1, x_2, x_3) \mapsto (x_1 + x_3, x_1 + x_2)$, $f_3 : (x_1, x_2, x_3) \mapsto (2x_2, x_1)$.

S5.10 Let K' be a subfield of the field K, V be a K'-vector space and W be a K-vector space. Then W is a K'-vector space in a natural way. With this vector space structure $\operatorname{Hom}_{K'}(V, W)$ is even a K-subspace of W^V.

†**S5.11** (Characters) In this exercise we give generalization of the Lemma of Dedekind-Artin (see Exercise 5.5). Let M be a monoid and et K be a division ring (not necessarily commutative ring with $K^\times = K \smallsetminus \{0\}$, i. e., every non-zero element have multiplicative inverse).

(a) Let $\varphi_1, \ldots, \varphi_r \in K^M$ be characters of M with values in K which are linearly independent over K. If a linear combination $\varphi = a_1\varphi_1 + \cdots + a_r\varphi_r$ with coefficients $a_1, \ldots, a_r \in K$ is also a character of M with values in K, then $\varphi = \kappa_{a_i}\varphi_i$ for every i with $a_i \neq 0$. (**Hint :** Note that: for all $x, y \in M$, on one side, we have

$$\varphi(xy) = a_1\varphi_1(xy) + \cdots + a_r\varphi_r(xy) = a_1\varphi_1(x)\,\varphi_1(y) + \cdots + a_r\varphi_r(x)\,\varphi_r(y)$$

and the other-side

$$\varphi(xy) = \varphi(x)\,\varphi(y) = a_1\varphi(x)\,\varphi_1(y) + \cdots + a_r\varphi(x)\,\varphi_r(y)\,.)$$

(b) Let $M = G$ be a group. Then a character $G \to K$ is then a group homomorphism $G \to K^\times$ and the group $\operatorname{Hom}(G, K^\times)$ of characters is a subgroups of $(K^\times)^G$. *If G is finite and $\chi : G \to K^\times$ is not a trivial character, then $\sum_{x \in G} \chi(x) = 0$.* (**Hint :** If $y \in G$ is an element with $\chi(y) \neq 1_K$, then $\sum_{x \in G} \chi(x) = \sum_{x \in G} \chi(xy) = \left(\sum_{x \in G} \chi(x)\right) \chi(y)$, and hence $\sum_{x \in G} \chi(x) = 0$, since $\chi(y) \neq 1$. — **Remark :** The group $\operatorname{Hom}(G, \mathbb{C}^\times)$ of characters with values in the field $\mathbb{C}$ is called the character group of G and is denoted by $\hat{G}$. This group plays an important roll is the study of abelian groups.)

S5.12 Some simple applications of the Lemma of Dedekind-Artin (see Exercise 5.5.)

(a) Let K be a field. The maps $K \to K$, $t \mapsto t^n$, $n \in \mathbb{N}$, are the only polynomial maps of K into itself which are also characters of the multiplicative monoid of K with values in K. More generally : The functions $t \mapsto t^n$, $n \in \mathbb{Z}$, are the only group homomorphisms of $K^\times \to K^\times$, which are also rational functions on $K^\times$. (**Hint :** The case that K is finite should be treated separately ; in this case use the fact that the multiplicative group $K^\times$ is cyclic.)

(b) The functions $t \mapsto \exp at$, $a \in \mathbb{C}$, of $\mathbb{R}$ in $\mathbb{C}$ are linearly independent over $\mathbb{C}$.

(c) Let K be a field. The sequences $(a^\nu)_{\nu \in \mathbb{N}}$, $a \in K$, are linearly independent over K. In particular, the $\mathbb{R}$-vector space $\mathbb{R}^{\mathbb{N}}$ is uncountable dimensional. (**Remark :** See also Exercise 3.4 (a).)

[1] A polynomial $P(t) = \sum_{i=0}^n a_i t^i \in K[t]$ of degree n over a field K is called a monic polynomial if the leading co-efficient $a_n = 1$.

†**S5.13** (Continuous characters of $\mathbb{R}$ and $\mathbb{C}$) The aim of this Supplement is to describe all continuous characters of the fields of real and complex numbers.

(a) Every continuous character $\chi : \mathbb{R}^\times \to \mathbb{R}^\times$ is either of the form $x \mapsto |x|^\beta$ or of the form $x \mapsto |x|^\beta \operatorname{Sign} x$ with a (uniquely determined) $\beta \in \mathbb{R}$.

(b) Every continuous character $\chi : \mathbb{C}^\times \to \mathbb{C}^\times$ is of the form $z \mapsto |z|^\alpha z^n$ with (uniquely determined) elements $\alpha \in \mathbb{C}$ and $n \in \mathbb{Z}$.

(c) The functions $z \mapsto z^n$, $n \in \mathbb{Z}$, are the only continuous endomorphisms of the circle-group $\mathrm{S}^1 := \{z \in \mathbb{C} \mid |z| = 1\}$. In particular, identity $\mathrm{S}^1 \to \mathrm{S}^1$, $z \mapsto z$ and the inverse-mapping $\mathrm{S}^1 \to \mathrm{S}^1$, $z \mapsto z^{-1}$ are the only continuous automorphisms of S^1. (**Hint :** Use parts (a) and (b) above.)

†**S5.14** Let S^1 be the circle-group $\{z \in \mathbb{C} \mid |z| = 1\}$.

(a) Every continuous character $\mathrm{S}^1 \to \mathbb{C}^\times$ and every complex-analytic character $\mathbb{C}^\times \to \mathbb{C}^\times$ is of the form $z \mapsto z^n$ with a unique $n \in \mathbb{Z}$.

(b) Every continuous group homomorphism $\mathbb{C}^\times \to \mathrm{S}^1$ is of the form $z \mapsto |z|^{-n+\mathrm{i}\gamma} z^n$ with a unique $\gamma \in \mathbb{R}$ and $n \in \mathbb{N}$.

(c) Every continuous character $(\mathbb{C}, +) \to \mathbb{C}^\times$ is of the form $z \mapsto e^{\alpha z} e^{\beta \bar{z}}$ with a unique $\alpha, \beta \in \mathbb{C}$. Further, its image is contained in S^1 respectively, in $\mathbb{R}^\times$, if and only if $\beta = -\bar{\alpha}$ respectively, $\beta = \bar{\alpha}$. Moreover, it is complex-analytic if and only if $\beta = 0$.

(d) Every continuous group homomorphism $\mathbb{C}^\times \to (\mathbb{C}, +)$ is of the form $z \mapsto \beta \ln |z|$ with a $\beta \in \mathbb{C}$. Every continuous group homomorphism $\mathrm{S}^1 \to (\mathbb{C}, +)$ and every complex-analytic group homomorphism $\mathbb{C}^\times \to (\mathbb{C}, +)$ is trivial.

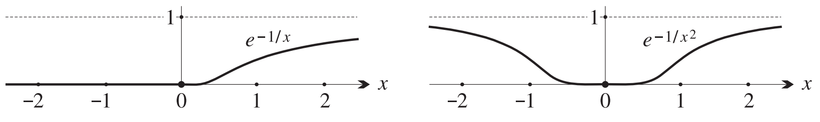
†**T5.15** Let $I \subseteq \mathbb{R}$ be an interval with more than one point and $a \in I$. Let $T_a : \mathrm{C}^\infty_{\mathbb{K}}(I) \to \mathbb{K}[\![t-a]\!]$ be the map which maps every function $f \in \mathrm{C}^\infty_{\mathbb{K}}(I)$ to its Taylor-series of f at a, i. e.,

$$T_a(f) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (t-a)^k .$$

Show that T_a is a $\mathbb{K}$-linear map of $\mathrm{C}^\infty_{\mathbb{K}}(I)$ in the space $\mathbb{K}[\![t-a]\!]$ of all (formal) power series in $(t-a)$ with coefficients in $\mathbb{K}$. The kernel of T_a is the space of all plate functions at a. Further, show that T_a is surjective.

(**Remarks :** The kernel of T_a is the space of all so-called plate functions at a.[2] — An infinitely many times differentiable function $f : I \to \mathbb{C}$ is called plate at point $a \in I$, if $f^{(n)}(a) = 0$ for all $n \in \mathbb{N}$. There are functions which are plate at a point, but are not identically zero in any neighbourhood of this point. Such a function cannot be analytic; for example, the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} e^{-1/x}, & \text{if } x > 0, \\ 0, & \text{if } x \le 0. \end{cases}$$



is infinitely many times differentiable and it is plate at 0.
Furthermore, T_a is surjective. This is immediate from the following classical theorem[3] of real analysis which is proved in 1895 by the French mathematician Borel, Émile Félix Édouard-Justin (1871-1956) in his PhD thesis.)

[2]Let $f : D \to \mathbb{C}$ be an *analytic* function on an interval $D \subseteq \mathbb{R}$ or a domain $D \subseteq \mathbb{C}$. If the derivatives $f^{(n)}(a)$ of f at a point $a \in D$ are zero, then by the *Taylor's formula* the function f vanishes in a neighbourhood of a and hence by the *identity theorem* f is identically 0 on the whole D. The analogous result does *not* hold for infinitely many times differentiable functions defined on an interval $I \subseteq \mathbb{R}$.

[3]**Theorem**(Borel) *For every sequence a_n, $n \in \mathbb{N}$, of real or complex numbers there exists an infinitely many times differentiable function f on $\mathbb{R}$ with values in $\mathbb{R}$ resp. $\mathbb{C}$ such that for all $n \in \mathbb{N}$ gilt: $f^{(n)}(0) = a_n$.*

A proof of Borel's theorem require a construction of so-called hat-functions if it satisfies properties stated in the following theorem : *Let $a, a', b', b \in \mathbb{R}$ with $a < a' < b' < b$. Then there exists an infinitely many times differentiable function $h : \mathbb{R} \to \mathbb{R}$ such that $h(t) = 0$ for $t \notin [a, b]$, $h(t) = 1$ for $t \in [a', b']$ and $0 < h(t) < 1$ otherwise.*

S5.16 (Algebras and Algebra homomorphisms) Let K be a field (or, moregenerally, a commutative ring).

(a) (Algebra over K) A vector space A over K together with a multiplication (need not be commutative) $A \times A \to A$, $(x, y) \mapsto xy$, is called an *algebra over* K, or a K-*algebra* if the following compatibility conditions hold :

(1) A is a ring with the vector space addition and the given multiplication.
(2) For all $a, b \in K$ and all $x, y \in A$, we have : $(ax)(by) = (ab)(xy)$.

(b) (Algebra-Homomorphisms) If A and B are two K-algebras, then a map $\varphi : A \to B$ is called a K-*algebra homomorphism* if :

(1) φ is a K-vector space homomorphism.
(2) φ is compatible with the multiplications on A and B, i. e., $\varphi(xy) = \varphi(x)\,\varphi(y)$ for all $x, y \in A$ and moreover, $\varphi(1_A) = 1_B$.

(c) Every K-algebra homomorphism is, in particular, a ring homomorphism.

(d) Let V be a K-vector space. Then $\mathrm{End}_K V$ is a K-Algebra and its unit group $(\mathrm{End}_K V)^\times$ is the automorphism-group $\mathrm{Aut}_K V$ von V.

S5.17 (Function-Algebras) An important class of (commutative) algebras is the class of function-algebras. For an arbitrary field K and an arbitrary set D, the set K^D of all K-valued functions on D is a commutative K-algebra in a natural way and the *substitution maps* $K^D \to K$, $x \mapsto x(t_0)$, for a fixed $t_0 \in D$, are K-algebra-homomorphisms. All examples of subspaces given in Supplement S2.10 and Exercise 2.1 (a) are even subalgebras of the algebra of the type K^D. There by a subset A' of a K-algebra A is called a (K-)*subalgebra of* A, if A' is a K-subspace as well as a subring of A.

S5.18 Let A be a K-Algebra. The map $A \to \mathrm{End}_K A$, $\lambda : x \mapsto \lambda_x$ (where λ_x is the left-multiplication by x) is an injective K-algebra-homomorphism of A in $\mathrm{End}_K(A)$. Therefore, every K-algebra A is (up to isomorphism) a subalgebra of the endomorphism-algebra of a K-vector space V. Moreover, if A has finite dimension n, then one can also choose V of dimension n.

S5.19 Let K be a field.

(a) Every 1-dimensional K-algebra is isomorphic to K.

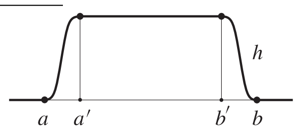
(b) Every two-dimensional K-algebra A has a basis of the form $1, x$ and hence it is commutative. The square x^2 is a linear combination $x^2 = \alpha + \beta x$ of 1 and x, and using this equation the multiplication in A is uniquely determined. (Typical Example: $\mathbb{C}$ with the basis $\mathbb{R}$-basis $1, \mathrm{i}$ and the equation $\mathrm{i}^2 = -1$.) The trivial subalgebras $K = K \cdot 1_A$ and A are the only subalgebras of A.

S5.20 Let A be a K-algebra and $x \in A$. The smallest K-subalgebra of A, containing x, is the subalgebra $K[x] := \sum_{i \in \mathbb{N}} Kx^i$ of all linear combinations of the powers x^i, $i \in \mathbb{N}$, of x. Show that :

(a) The K-subalgebra $K[x]$ is a finite dimensional K-algebra if and only if the powers x^i, $i \in \mathbb{N}$, linearly dependent over K.

(b) If $K[x]$ is finite dimensional and $\mathrm{Dim}_K K[x] = n$, then $1, x, \ldots, x^{n-1}$ a K-vector space basis of $K[x]$. In this case x is called *algebraic* over K (*of degree* n). If $K[x]$ is infinite dimensional, then x is called *transcendental over* K. If A is finite dimensional with $\mathrm{Dim}_K A = m$, then every element of A is algebraic over K of degree $\leq m$.

S5.21 Let I be a set. Show that:



Hat-functions are very useful for many constructions in analysis.

(a) The K-algebra K^I is c y c l i c or m o n o g e n i c, i. e., $K^I = K[x]$ for some $x \in K^I$ if and only if I finite and the map $x : I \to K$ injective.

(b) A map $x \in K^I$ is algebraic over K (see Supplement S5.20 (b)) if and only if x attains only finitely many values. Moreover, in this case the degree of x over K is equal to the number of elements $|x(I)|$ of these values.

S5.22 Let I be a finite set and A be a K-subalgebra of the function-algebra K^I. Show that $A = K^I$ if and only if A separates the points of I, i. e., if for every $i, j \in I$ with $i \neq j$, there exists an element $x \in A$ such that $x(i) \neq x(j)$. Using this result once again prove the assertion in Supplement S5.20 (a). (**Hint :** Suppose that A separates the points. Then, for every fixed $i \in I$ and for each $j \neq i$ choose $x_j \in A$ such that $a_j := x_j(j) \neq x_j(i)$. Then $\prod_{j \neq i}(x_j - a_j) \in A$ is a function, which vanishes on $I - \{i\}$ and takes the value $\neq 0$ at i.)

S5.23 Let I be a finite set.

(a) For every K-subalgebra A of K^I, the relation R_A on I, defined by $(i, j) \in R_A$ if and only if $f(i) = f(j)$ for every $f \in A$, is an equivalence relation on I.

(b) For every equivalence relation R on I, $A_R := \{f \in K^I \mid f$ constant on the equivalence classes of $R\}$ is a K-subalgebra of K^I. (The indicator functions e_J of the equivalence classes J form a K-basis of A_R.)

(c) Show that the maps $A \mapsto R_A$ and $R \mapsto A_R$ are inverse-maps from the set of all K-subalgebras of K^I onto the set of all equivalence relations on I. In particular, the number of K-subalgebras of K^I is equal to the Bell's number $\beta_{|I|}$. (**Hint :** Apply Supplement S5.22.)

S5.24 (T r i g o n o m e t r i c P o l y n o m i a l s) Let $\omega \in \mathbb{R}_+^\times$ be fixed. Then the $\mathbb{C}$-subspace $\sum_{n\in\mathbb{Z}} \mathbb{C}e^{i\omega nt}$ is a $\mathbb{C}$-subalgebra of $\mathrm{C}^{\omega}_{\mathbb{C}}(\mathbb{R})$. It is the smallest $\mathbb{C}$-subalgebra $\mathbb{C}[\sin\omega t, \cos\omega t]$ of $\mathrm{C}^{\omega}_{\mathbb{C}}(\mathbb{R})$, containing the functions $\sin\omega t$ and $\cos\omega t$ and the functions $1; \sin n\omega t, \cos n\omega t, n \in \mathbb{N}^*$, form a $\mathbb{C}$-basis. These functions also form a $\mathbb{R}$-basis of the $\mathbb{R}$-subalgebra $\mathbb{R}[\sin\omega t, \cos\omega t]$ of the $\mathbb{R}$-valued functions in $\mathbb{C}[\sin\omega t, \cos\omega t]$. (The algebras $\mathbb{C}[\sin\omega t, \cos\omega t]$ and $\mathbb{R}[\sin\omega t, \cos\omega t]$ are called the algebras of the t r i g o n o m e t r i c p o l y n o m i a l s corresponding to the basic-frequency ω.)

S5.25 Let A be a K-algebra (not necessarily commutative) and $a \in A^\times$ be a unit in A. Then the map $A \to A$, $\kappa_a : x \mapsto axa^{-1}$ is an K-algebra-automorphism of A. This is called the c o n j u g a t i o n by a or the i n n e r a utomorphism by a. The map $a \mapsto \kappa_a$ from $A^\times$ into the group $\mathrm{Aut}_{K-\mathrm{alg}} A$ of the K-algebra-automorphisms of A is a group homomorphism with the kernel $A^\times \cap \mathrm{Z}(A) = \mathrm{Z}(A)^\times$, where $\mathrm{Z}(A)$ denote the c e n t e r of A, which is the K-subalgebra of those elements $a \in A$, which commute with *all* elements of A.

S5.26 Let K be a finite field with $|K| = q$. Show that the polynomials functions $K \to K$, $t \mapsto t^i$, $i = 0, \ldots, q-1$, form a K-basis of the K-algebra $\mathrm{Pol}_K(K)$ of all polynomial functions from K into K. Every function $K \to K$ is a polynomial function, i. e., $K^K = \mathrm{Pol}_K(K)$. (Note that $x^q = x$ for all $x \in K$, see Supplement S2.9, Exercise 2.4.)

S5.27 Let A be an algebra over an *infinite* field K which has only finitely many K-subalgebras. Then show that A is a finite dimensional K-vector space and is monogenic K-algebra, i. e., $A = K[x]$. In particular, A is commutative. (If $A_1, \ldots, A_n$ are proper K-subalgebras of A, then choose $x \in A \smallsetminus (\cup_{i=1}^n A_i)$, see Exercise 2.2.)