E0 219 Linear Algebra and Applications / August-December 2016

(ME, MSc. Ph. D. Programmes)

Download from : http://www.math.iisc.ernet.in/patil/courses/courses/Current Courses/...

Tel : +91-(0)80-2293 2239/(Maths Dept. 3212) **E-mails :** dppatil@csa.iisc.ernet.in / patil@math.iisc.ernet.in

Lectures : Monday and Wednesday ; 11:00–12:30 **Venue:** CSA, Lecture Hall (Room No. 117)

Corrections by : **Nikhil Gupta** (nikhil.gupta@csa.iisc.ernet.in ; **Lab No.: 303**) /
Vineet Nair (vineetn90@gmail.com ; **Lab No.: 303**) /
Rahul Gupta (rahul.gupta@csa.iisc.ernet.in ; **Lab No.: 224**) /
Sayantan Mukherjee (meghanamande@gmail.com ; **Lab No.: 253**) /
Palash Dey (palash@csa.iisc.ernet.in ; **Lab No.: 301 , 333 , 335**)

Midterms : **1-st Midterm :** Saturday, September 17, 2016; 15:00 – 17:00 **2-nd Midterm :** Saturday, October 22, 2016; 15:00 – 17:00

Final Examination : December ??, 2016, 09:00 – -12:00

Evaluation Weightage : Assignments : 20% **Midterms (Two) :** 30% **Final Examination :** 50%

Range of Marks for Grades (Total 100 Marks)						
	Grade S	**Grade A**	**Grade B**	**Grade C**	**Grade D**	**Grade F**
Marks-Range	> 90	76 — 90	61 — 75	46 — 60	35 — 45	< 35

	Grade A$^+$	**Grade A**	**Grade B$^+$**	**Grade B**	**Grade C**	**Grade D**	**Grade F**
Marks-Range	> 90	81 — 90	71 — 80	61 — 70	51 — 60	40 — 50	< 40

Supplement 8

Quotient spaces and Exact sequences[1]

To understand and appreciate the Supplements which are marked with the symbol † one may possibly require more mathematical maturity than one may have! These are steps towards applications to various other branches of mathematics, especially to analysis, number theory and Affine and Projective Geometry.

Participants may ignore these Supplements — altogether or in the first reading!!

In the following Supplements K denote a field and V denote a K-vector space.

S8.1 (Exact Sequences and Complexes) Let G', G, G'' be (additive) abelian groups and $g' : G' \to G$, $g : G \to G''$ be homomorphisms. Then the sequence

$$G' \xrightarrow{g'} G \xrightarrow{g} G''$$

is called a complex (or a zero-sequence), if $\operatorname{Im} g' \subseteq \operatorname{Ker} g$, i. e., $g\,g' = 0$. In this case the residue class group

$$H := \mathrm{H}(G' \xrightarrow{g'} G \xrightarrow{g} G'') := \operatorname{Ker} g / \operatorname{Im} g'$$

is called the homology (group) of the complex. If this group is 0, i. .e., if $\operatorname{Im} g' = \operatorname{Ker} g$, then the complex is or also the sequence is called exact. In the case of a complex $\operatorname{Ker} g$ is called the group of the cycles and $\operatorname{Im} g'$ is called the group of the boundaries.[2] These groups are usually denoted by Z and B, respectively.[3]. Therefore $H = Z/B$.

A sequence

$$G_\bullet : \quad \cdots \longrightarrow G_{i+1} \xrightarrow{g_{i+1}} G_i \xrightarrow{g_i} G_{i-1} \longrightarrow \cdots$$

of abelian groups and homomorphisms is called a complex (or a zero-sequence), if for every $i \in \mathbb{Z}$, for which g_{i+1} and g_i are defined, the sequence $G_{i+1} \xrightarrow{g_{i+1}} G_i \xrightarrow{g_i} G_{i-1}$ is a complex. If $Z_i = \mathrm{Z}_i(G_\bullet)$ and $B_i = \mathrm{B}_i(G_\bullet)$ are the groups of the cycles and boundaries at the position i, respectively, then the quotient group

$$H_i = \mathrm{H}_i(G_\bullet) := \mathrm{Z}_i(G_\bullet)/\mathrm{B}_i(G_\bullet) = Z_i/B_i = \operatorname{Ker} g_i / \operatorname{Im} g_{i+1}$$

[1] Exact sequences and – more generally, Complexes are useful tools for well-arranged convenient description of recurring deductions in connection with homomorphisms of groups and in particular of vector spaces.

[2] These notation and terminology have originated in the algebraic topology.

[3] B for Boundary.

is called the i-th homomology (group) of the complex $G_\bullet$. If $H_i = 0$, then the complex $G_\bullet$ is called exact at the position i. The complex $G_\bullet$ is called exact if all of its homology group vanish, i. e., it is exact at every position.

Remark : These concepts and results can be carried over to the sequences of vector spaces and vector space homomorphisms (and generally to modules and module homomorphisms).

(a) Let $f : G \to F$ be a homomorphism of abelian groups. Then the homology of the complex $0 \to G \xrightarrow{f} F$ (where $0 \to G$ is the zero-homomorphism) is $\operatorname{Ker} f$. *This complex is exact if and only if f injective.* The homology of the complex $G \xrightarrow{f} F \to 0$ is the Cokernel $\operatorname{Coker} f := F/\operatorname{Im} f$ of f. *This complex is exact if and only if f is surjective.*

Altogether, the complex $0 \to G \xrightarrow{f} F \to 0$ is exact if and only if f is an isomorphism.
More generally, $f : G \to F$ defined so-called exact four-sequence

$$0 \longrightarrow \operatorname{Ker} f \xrightarrow{\iota} G \xrightarrow{f} F \xrightarrow{\pi} \operatorname{Coker} f \longrightarrow 0\,,$$

where ι is the canonical injection of $\operatorname{Ker} f \subseteq G$ in G and π is the canonical projection of F onto $\operatorname{Coker} f = F/\operatorname{Im} f$.

(b) (Short exact (three-term) sequence) A sequence

$$0 \longrightarrow G' \xrightarrow{g'} G \xrightarrow{g} G'' \longrightarrow 0$$

is, obviously, exact if and only if g' is injective and g is surjective and $U := \operatorname{Ker} g = \operatorname{Im} g'$. In this case g' induces an isomorphism $G' \to U$ and g induces an isomorphism $G/U \to G''$. Such an exact sequence is called a short exact (three-term)-sequence.

Every subgroup U of an abelian group G, is in the following short exact sequence with the cannonical homomorhisms ι and π :

$$0 \longrightarrow U \xrightarrow{\iota} G \xrightarrow{\pi} G/U \longrightarrow 0\,.$$

Moreover, one can also consider the short exact sequences of not necessarily abelian (multiplicative) groups

$$1 \longrightarrow G' \xrightarrow{g'} G \xrightarrow{g} G'' \longrightarrow 1\,,$$

if the above conditions are fulfilled.[4] Then $\operatorname{Ker} g = \operatorname{Im} g' \cong G'$ is necessarily a normal subgroup of G.

S8.2 (Homomorphisms of complexes) Let

$$\begin{array}{lll} G_\bullet & : & \cdots \longrightarrow G_{i+1} \xrightarrow{g_{i+1}} G_i \xrightarrow{g_i} G_{i-1} \longrightarrow \cdots \\ F_\bullet & : & \cdots \longrightarrow F_{i+1} \xrightarrow{f_{i+1}} F_i \xrightarrow{f_i} F_{i-1} \longrightarrow \cdots \end{array}$$

be two complexes which are defined for the same indices $i \in \mathbb{Z}$. A family $h_\bullet$ of homomorphisms $h_i : G_i \to F_i$, $i \in \mathbb{Z}$, is called a homomorphism of complexes if all the diagrams

$$\begin{array}{ccc} G_i & \xrightarrow{g_i} & G_{i-1} \\ {\scriptstyle h_i}\downarrow & & \downarrow{\scriptstyle h_{i-1}} \\ F_i & \xrightarrow{f_i} & F_{i-1} \end{array}$$

are commutative, that is, $h_{i-1} g_i = f_i h_i$ for all $i \in \mathbb{Z}$. In this case, obviously, h_i maps the cycle-groups $\mathrm{Z}_i(G_\bullet) = \operatorname{Ker} g_i$ into the cycle-groups $\mathrm{Z}_i(F_\bullet) = \operatorname{Ker} f_i$ and (if h_{i+1} is still defined) also the boundary-groups $\mathrm{B}_i(G_\bullet) = \operatorname{Im} g_{i+1}$ into the boundary-groups $\mathrm{B}_i(F_\bullet) = \operatorname{Im} f_{i+1}$, abd hence induce a homomorphism

$$\mathrm{H}_i(h_\bullet) : \mathrm{H}_i(G_\bullet) \longrightarrow \mathrm{H}_i(F_\bullet)\,.$$

[4] We denote the trivial (multiplicative) group by 1.

(a) (Snake-Lemma) Let

$$\begin{array}{ccccccc} & & G' & \xrightarrow{g'} & G & \xrightarrow{g} & G'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h''} & & \\ 0 & \longrightarrow & F' & \xrightarrow{f'} & F & \xrightarrow{f} & F'' & & \end{array}$$

be a commutative diagram with exact rows. Then the complexes

$$\mathrm{Ker}\, h' \xrightarrow{g'} \mathrm{Ker}\, h \xrightarrow{g} \mathrm{Ker}\, h''\,, \qquad \mathrm{Coker}\, h' \xrightarrow{\overline{f'}} \mathrm{Coker}\, h \xrightarrow{\overline{f}} \mathrm{Coker}\, h''\,,$$

are exact. More importantly, there is a canonical homomorphism $\delta : \mathrm{Ker}\, h'' \longrightarrow \mathrm{Coker}\, h'$, which connects both these exact sequences into so-called exact Ker-Coker-sequence[5]

$$\mathrm{Ker}\, h' \xrightarrow{g'} \mathrm{Ker}\, h \xrightarrow{g} \mathrm{Ker}\, h''\,, \overset{\delta}{- - - - - - - - \rightarrow} \mathrm{Coker}\, h' \xrightarrow{\overline{f'}} \mathrm{Coker}\, h \xrightarrow{\overline{f}} \mathrm{Coker}\, h''\,,$$

The homomorphism δ is also known as the connecteing-homomorphism.

(**Proof :** The connecting-homomorphism is defined as follows : Let $x'' \in \mathrm{Ker}\, h''$. Since g is surjective, there exists a $x \in G$ with $g(x) = x''$. Then $fh(x) = h''g(x) = h''(x'') = 0$, i.e., $h(x) \in \mathrm{Ker}\, f = \mathrm{Im}\, f'$ and hence $h(x) = f'(y')$ with (uniquely determined) $y' \in F'$. One can then define $\delta(x'') := \overline{y'} \in \mathrm{Coker}\, h' = F'/\mathrm{Im}\, h'$. The image $\delta(x'')$ does not depend on the choice of the pre-image x of x'' : Namely, if $g(\widetilde{x}) = x''$ also, then $x - \widetilde{x} \in \mathrm{Ker}\, g = \mathrm{Im}\, g'$, i.e., $x - \widetilde{x} = g'(x')$ and for $\widetilde{y'} \in F'$ with $h(\widetilde{x}) = f'(\widetilde{y'})$ it follows that $y' - \widetilde{y'} = h'(x')$, and hence $\overline{y'} = \overline{\widetilde{y'}}$ in $F'/\mathrm{Im}\, h'$.

It is easy to check that δ is a homomorphism and that the given sequence is exact at the positions $\mathrm{Ker}\, h''$ and $\mathrm{Coker}\, h'$. Similar to the "diagram chasing" as done in the above prof of independence in the definition o δ, one can check the exactness at the other positions. If g' is injective (resp. if f surjective), then naturally, $\mathrm{Ker}\, h' \longrightarrow \mathrm{Ker}\, h$ is also injective (resp. $\mathrm{Coker}\, h \longrightarrow \mathrm{Coker}\, h''$ is surjective).)

(b) The following assertion is used very often. *Let*

$$0 \longrightarrow V_n \xrightarrow{f_n} V_{n-1} \longrightarrow \cdots \longrightarrow V_1 \xrightarrow{f_1} V_0 \longrightarrow 0$$

is an exact sequence of finite dimensional K-vector spaces. Then the alternating sum of dimensions vanishes, i.e.,

$$\sum_{i=0}^{n} (-1)^i \mathrm{Dim}_K V_i = 0\,.$$

(**Proof :** By induction on n. The cases $n = 0$ and $n = 1$ are trivial, in the case $n = 2$, since $V_0 = \mathrm{Im}\, f_1$ and $V_2 \cong \mathrm{Im}\, f_2 = \mathrm{Ker}\, f_1$, follows by applying the Rank Theorem to f_1. For $n \geq 3$, we apply induction hypothesis to the exact seqeunces :

$$0 \longrightarrow V_n \xrightarrow{f_n} V_{n-1} \longrightarrow \cdots \longrightarrow V_2 \xrightarrow{f_2} \mathrm{Bild}\, f_2 \longrightarrow 0\,, \quad \text{and} \quad 0 \longrightarrow \mathrm{Bild}\, f_2 \longrightarrow V_1 \xrightarrow{f_1} V_0 \longrightarrow 0$$

and note that by induction hypothesis, we have

$$\sum_{i=2}^{n} (-1)^{i-1} \mathrm{Dim}_K V_i + \mathrm{Dim}_K \mathrm{Im}\, f_2 = 0\,, \quad \text{and} \quad \mathrm{Dim}_K \mathrm{Im}\, f_2 - \mathrm{Dim}_K V_1 + \mathrm{Dim}_K V_0 = 0\,.)$$

S8.3 (Functors $\mathrm{Hom}_K(-X)$ and $\mathrm{Hom}_K(X,-)$) An important aspect in the theory of vector spaces is that exact sequences remain exact after passing them to the homomorphism spaces. More precisely :

Let $f : V \to W$ be a homomorphism of K-vector spaces and X be another K-vector space. For every homomorphism $h : W \to X$, the composition hf is a homomorphism $V \to X$. This defines a K-vector space homomorphism

$$\mathrm{Hom}_K(W,X) \longrightarrow \mathrm{Hom}_K(V,X)$$

which is denoted by $\mathrm{Hom}_K(f,X)$. Analogously, the map $g \mapsto fg$ defines a homomorphism

$$\mathrm{Hom}_K(X,V) \longrightarrow \mathrm{Hom}_K(X,W)$$

which is denoted by $\mathrm{Hom}_K(X,f)$. In the case $X = K$, the map $\mathrm{Hom}_K(f,K)$ is nothing but the map which associates f to its dual homomorphism $f^* : W^* \to V^*$ (and using the canonical identification

[5] This exact sequence explains the name "Snake-Lemma".

of $\mathrm{Hom}_K(K,V)$ with V and of $\mathrm{Hom}_K(K,W)$ with W, the map $\mathrm{Hom}(K,f)$ is the map f it self, see Supplement S5.8. With this we have :

Let $V' \xrightarrow{f'} V \xrightarrow{f} V''$ be an exact sequence of K-vector spaces and X be another K-vector space. Then the following corresponding sequences are also exact :

$$\mathrm{Hom}_K(V'',X) \longrightarrow \mathrm{Hom}_K(V,X) \longrightarrow \mathrm{Hom}_K(V',X) \,,$$
$$\mathrm{Hom}_K(X,V') \longrightarrow \mathrm{Hom}_K(X,V) \longrightarrow \mathrm{Hom}_K(X,V'') \,.$$

(**Proof :**

•)

S8.4 Let $f:V \to W$ be a homomorphism of K-vector spaces.

(a) Dualising the canonical short exact sequences

$$0 \longrightarrow \mathrm{Ker}\, f \longrightarrow V \to \mathrm{Im}\, f \longrightarrow 0 \qquad \text{and} \qquad 0 \longrightarrow \mathrm{Im}\, f \longrightarrow W \to \mathrm{Coker}\, f \longrightarrow 0$$

we get the short exact sequences

$$0 \longrightarrow (\mathrm{Im}\, f)^* \longrightarrow V^* \longrightarrow (\mathrm{Ker}\, f)^* \longrightarrow 0 \quad \text{and} \quad 0 \to (\mathrm{Coker}\, f)^* \longrightarrow W^* \longrightarrow (\mathrm{Im}\, f)^* \longrightarrow 0$$

and in particular, a canonical isomorphism $(\mathrm{Im}\, f)^* \cong \mathrm{Im}\, f^*$. (Since the composition of the surjective $W^* \to (\mathrm{Im}\, f)^*$ map and the injective map $(\mathrm{Im}\, f^*) \to V^*$ is the dual map f^*.)

(b) The Rank f is finite if and only if Rank f^* is finite. In this case, the equality $\mathrm{Rank}\, f = \mathrm{Rank}\, f^*$. See Theorem 5.G.19 and the remark after that. From the 4-term exact sequence

$$0 \to \mathrm{Ker}\, f \longrightarrow V \xrightarrow{f} W \longrightarrow \mathrm{Coker}\, f \longrightarrow 0$$

the exactness of the following 4-term sequence follows directly

$$0 \longrightarrow (\mathrm{Coker}\, f)^* \longrightarrow W^* \xrightarrow{f^*} V^* \longrightarrow (\mathrm{Ker}\, f)^* \longrightarrow 0$$

and hence canonical isomorphisms

$$(\mathrm{Ker}\, f)^* \cong \mathrm{Coker}\, f^* \,, \quad (\mathrm{Coker}\, f)^* \cong \mathrm{Ker}\, f^* \,,$$

further, the characterisations of $\mathrm{Im}\, f^*$ as the space of linear forms on V, which vanish on the $\mathrm{Ker}\, f$ (whereas $\mathrm{Ker}\, f^*$ is the space of linear forms on W, which vanish on $\mathrm{Im}\, f$).

S8.5 (Cohomology) Occasionally, the groups or vector spaces of a complexes are denoted by upper indices, then the numbering is increasing, and hence

$$G^\bullet \,:\, \cdots \longrightarrow G^{i-1} \xrightarrow{g^{i-1}} G^i \xrightarrow{g^i} G^{i+1} \longrightarrow \cdots \,.$$

Instead of cycles and boundaries, one use the terms cocycles and coboundaries, and

$$H^i = \mathrm{H}^i(G^\bullet) := \mathrm{Z}^i(G^\bullet)/\mathrm{B}^i(G^\bullet) = \mathrm{Ker}\, g^i / \mathrm{Im}\, g^{i-1}$$

is called the i-th cohomology(group) of the complex $G^\bullet$.

S8.6 (Meyer-Vietoris-sequences) Let H and F be subgroups of the abelian group G. Then the so-called Meyer-Vietoris-sequences

$$0 \longrightarrow H \cap F \xrightarrow{f} H \oplus F \xrightarrow{g} H + F \longrightarrow 0$$

with $f(x) = (x,-x)$ and $g(y,z) = y+z$ and

$$0 \longrightarrow G/(H \cap F) \xrightarrow{h} (G/H) \oplus (G/F) \xrightarrow{k} G/(H+F) \longrightarrow 0$$

with $h(\overline{x}) = (\overline{x}, -\overline{x})$ and $k(\overline{y},\overline{z}) = \overline{y+z}$ are exact.

S8.7 (Five-Lemma) Suppose that in the following commutative diagram

$$\begin{array}{ccccccccc}
G_5 & \xrightarrow{g_5} & G_4 & \xrightarrow{g_4} & G_3 & \xrightarrow{g_3} & G_2 & \xrightarrow{g_2} & G_1 \\
{\scriptstyle h_5}\downarrow & & {\scriptstyle h_4}\downarrow & & {\scriptstyle h_3}\downarrow & & {\scriptstyle h_2}\downarrow & & {\scriptstyle h_1}\downarrow \\
F_5 & \xrightarrow{f_5} & F_4 & \xrightarrow{f_4} & F_3 & \xrightarrow{f_3} & F_2 & \xrightarrow{f_2} & F_1
\end{array}$$

of abelian groups rows are exact. Then :

(a) if h_2 and h_4 are injective, then h_3 is also injective.
(b) if h_2 and h_4 are surjective and h_1 injective, then h_3 is surjective.
(c) if h_1, h_2, h_4, h_5 are bijective, then h_3 is also bijective.

(**Proof :** One can prove these assertions by the standard technique of "diagram- chasing", but we give a proof using Snake-Lemma (see Supplement S8.2.

(a) Since $h_3 g_4 = f_4 h_4$, $h_3(\operatorname{Im} g_4) \subseteq \operatorname{Im} f_4$ and hence h_3 induces a homomorphism $h_3' : \operatorname{Im} g_4 \to \operatorname{Im} f_4$. Since h_5 is surjective, $h_5' := f_5 \circ h_5 : G_5 \to \operatorname{Im} f_5$ is also surjective and since h_2 is injective, the restriction $h_2' = h_2\,|_{\operatorname{Im} g_3} : \operatorname{Im} g_3 \to F_2$ is also injective. Let ι denote the canonical embedding, then from the given commutative diagram, we get the following two commutative diagrams with exact rows :

$$\begin{CD}
@. G_5 @>g_5>> G_4 @>h_4>> \operatorname{Im} g_4 @>>> 0\\
@. @V h_5' VV @V h_4 VV @V h_3' VV\\
0 @>>> \operatorname{Im} f_5 @>\iota>> F_4 @>f_4>> \operatorname{Im} f_4
\end{CD}$$

$$\begin{CD}
@. \operatorname{Im} g_4 @>\iota>> G_3 @>g_3>> \operatorname{Im} g_3 @>>> 0\\
@. @V h_3' VV @V h_3 VV @V h_2' VV\\
0 @>>> \operatorname{Im} f_4 @>\iota>> F_3 @>f_3>> F_2
\end{CD}$$

Now, we use Snake-Lemma (see Supplement S8.2) and consider the exact Ker-Coker sequences (with connecting homomorphism δ) :

$$\operatorname{Ker} h_4 \longrightarrow \operatorname{Ker} h_3' \xrightarrow{\ \delta\ } \operatorname{Coker} h_5' \ ; \qquad \operatorname{Ker} h_3' \longrightarrow \operatorname{Ker} h_3 \longrightarrow \operatorname{Ker} h_2' .$$

Therefore, $\operatorname{Ker} h_4 = 0$, since h_4 is injective and $\operatorname{Coker} h_5' = 0$, since h_5' is surjective. Further, since the sequence is exact, one must have $\operatorname{Ker} h_3' = 0$. Since h_2' is injective, it follows $\operatorname{Ker} h_2' = 0$ and the second exact sequence shows that $\operatorname{Ker} h_3 = 0$, i. e., h_3 is injective.

(b) Since $h_2 g_3 = f_3 h_3$, $h_3(\operatorname{Ker} g_3) \subseteq \operatorname{Ker} f_3$ and hence h_3 induces a homomorphism $\overline{h}_3 : G_3/\operatorname{Ker} g_3 \to F_3/\operatorname{Ker} f_3$. Let p denote the canonical projection on the residue class groups. Then $\overline{h}_4 := p \circ h : G_4 \to F_4/\operatorname{Ker} f_4$ is surjective, since h_4 is surjective. Moreover, let $h_1' = h_1\,|_{\operatorname{Im} g_2}$ which is a restriction of h_1 is injective, since h_1 is injective. Further, $\overline{f}_4$ resp. $\overline{g}_3$ denote the maps induced by f_4 resp. g_3. Now, from the given commutative diagram, we get the following two commutative diagrams with exact rows :

$$\begin{CD}
@. G_4 @>g_4>> G_3 @>p>> G_3/\operatorname{Ker} g_3 @>>> 0\\
@. @V \overline{h}_4 VV @V h_3 VV @V \overline{h}_3 VV\\
0 @>>> F_4/\operatorname{Ker} f_4 @>\overline{f}_4>> F_3 @>p>> F_3/\operatorname{Ker} f_3
\end{CD}$$

$$\begin{CD}
@. G_3/\operatorname{Ker} g_3 @>\overline{g}_3>> G_2 @>g_2>> \operatorname{Im} g_2 @>>> 0\\
@. @V \overline{h}_3 VV @V h_2 VV @V h_1' VV\\
0 @>>> F_3/\operatorname{Ker} f_3 @>\overline{f}_3>> F_2 @>f_2>> F_1
\end{CD}$$

Now, we use Snake-Lemma (see Supplement S8.2) and consider the exact Ker-Coker sequences (with connecting homomorphism δ) :

$$\operatorname{Coker} \overline{h}_4 \longrightarrow \operatorname{Coker} h_3 \xrightarrow{\ \delta\ } \operatorname{Coker} \overline{h}_3 \ ; \qquad \operatorname{Ker} h_1' \longrightarrow \operatorname{Coker} \overline{h}_3 \longrightarrow \operatorname{Coker} h_2 .$$

In the second exact sequence $\operatorname{Ker} h_1' = 0$, since h_1' is injective and $\operatorname{Coker} h_2 = 0$, since h_2 is surjective. Further, since the sequence is exact, one must have $\operatorname{Coker} \overline{h}_3 = 0$. In the first exact sequence $\operatorname{Coker} \overline{h}_4 = 0$, since $\overline{h}_4$ is surjective and hence $\operatorname{Coker} h_3 = 0$, by the exactness of the seqeunce i. e., h_3 is surjective. •)

S8.8 (Euler-Poincaré-Characteristic) Let

$$V_\bullet \ : \quad 0 \longrightarrow V_n \xrightarrow{\ f_n\ } V_{n-1} \longrightarrow \cdots \longrightarrow V_1 \xrightarrow{\ f_1\ } V_0 \longrightarrow 0$$

be a complex of finite dimensional K-vector spaces. If $H_0, H_1, \ldots, H_{n-1}, H_n$, are homology spaces of $V_\bullet$, then (generalisation of Example ???) we have

$$\sum_{i=0}^{n} (-1)^i \operatorname{Dim}_K H_i = \sum_{i=0}^{n} (-1)^i \operatorname{Dim}_K V_i .$$

(**Remark :** This alternating sum is known as the Euler-Poincaré-Characteristic of the complex $V_\bullet$ and is denoted by $\chi(V_\bullet)$. One can already define it if the homology spaces H_i, $i = 0, \ldots, n$, are finite dimensional.

Analogously, for a complex of finite abelian groups $G_\bullet$: $0 \longrightarrow G_n \longrightarrow \cdots \longrightarrow G_0 \longrightarrow 0$ with homology groups $H_0, \ldots, H_n$, one has
$$\prod_{i=0}^{n} |H_i|^{(-1)^i} = \prod_{i=0}^{n} |G_i|^{(-1)^i} .)$$

S8.9 (Index of a linear map) If the kernel $\operatorname{Ker} f$ and the cokernel $\operatorname{Coker} f$ of a K-linear map $f : V \to W$ are finite dimensional, then we say that f have an index, and define
$$\operatorname{Ind} f := \operatorname{Dim}_K \operatorname{Ker} f - \operatorname{Dim}_K \operatorname{Coker} f$$
(Therefore $-\operatorname{Ind} f$ is the Euler-Poincaré-Characteristic of the complex $0 \longrightarrow V \xrightarrow{f} W \longrightarrow 0$.)

(a) If V and W are finite dimensional, then $\operatorname{Ind} f = \operatorname{Dim}_K V - \operatorname{Dim}_K W$.

(b) Let
$$\begin{array}{ccccccccc}
0 & \longrightarrow & V_n & \longrightarrow & \cdots & \longrightarrow & V_0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_n} & & & & \downarrow{\scriptstyle f_0} & & \\
0 & \longrightarrow & W_n & \longrightarrow & \cdots & \longrightarrow & W_0 & \longrightarrow & 0
\end{array}$$
be a commutative diagram of K-vector spaces and K-linear maps with exact rows. If all the linear maps $h_0, h_1, \ldots, h_n$ except one of them are of finite index, then all these linear maps are of finite index and $\sum_{i=0}^{n} (-1)^i \operatorname{Ind} h_i = 0$. (**Hint :** By induction on n. In the case $n = 2$, use the Snake-Lemma Supplement S8.2.)

(c) If $f : V \to W$ and $g : W \to X$ have index, then the composition $gf : V \to X$ also have index and $\operatorname{Ind} gf = \operatorname{Ind} g + \operatorname{Ind} f$. (**Hint :** One may consider the following commutative diagram with exact rows :
$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \operatorname{Ker} f & \longrightarrow & V & \xrightarrow{f} & W & \longrightarrow & \operatorname{Coker} f & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}_V} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \bar{g}} & & \\
0 & \longrightarrow & \operatorname{Ker} gf & \longrightarrow & V & \xrightarrow{gf} & X & \longrightarrow & \operatorname{Coker} gf & \longrightarrow & 0
\end{array} \quad .)$$

(d) If $f : V \to W$ have an index and if $g : V \to W$ have finite rank, then $f + g$ has index and $\operatorname{Ind}(f + g) = \operatorname{Ind} f$. (**Hint :** Define $U := \operatorname{Im} g$ and $(f, g)(x) := \big(f(x), g(x)\big)$ and consider the following commutative diagrams
$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & V & \xrightarrow{\mathrm{id}} & V & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle (f,g)} & & \downarrow{\scriptstyle f+g} & & \\
0 & \longrightarrow & U & \longrightarrow & W \oplus U & \longrightarrow & W & \longrightarrow & 0
\end{array}$$
$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & V & \xrightarrow{\mathrm{id}} & V & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle (f,g)} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & U & \longrightarrow & W \oplus U & \longrightarrow & W & \longrightarrow & 0 .)
\end{array}$$

(e) The K-linear map $f : V \to W$ has an index if and only if its dual map $f^* : W^* \to V^*$ has an index. In this case, $\operatorname{Ind} f^* = -\operatorname{Ind} f$. (**Hint :** see Supplement S8.4 (b).)

S8.10 If kernel and cokernel of a homomorphism $h : G \to F$ of abelian groups are finite, then we say that h has a Herbrand-quotient[6] and it is defined by
$$\mathrm{q}(h) := |\operatorname{Ker} h| / |\operatorname{Coker} h| .$$
(**Remark :** Note that analogy with the concept of the index in Supplement S8.9.)

(a) If G and F are finite, then $\mathrm{q}(h) = |G| / |F|$.

[6]The Herbrand quotient was invented by a French mathematician Jacques Herbrand (1908-1931). It has an important application in class field theory. Although he died at only 23 years of age, he was already considered one of "the greatest mathematicians of the younger generation" by his professors Helmut Hasse, and Richard Courant.

(b) Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_n & \longrightarrow & \cdots & \longrightarrow & G_0 & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle h_n} & & & & \big\downarrow{\scriptstyle h_0} & & \\ 0 & \longrightarrow & F_n & \longrightarrow & \cdots & \longrightarrow & F_0 & \longrightarrow & 0 \end{array}$$

be a commutative diagram of abelian groups and group homomorphisms. If all the homomorphisms $h_0, h_1, \ldots, h_n$ except one of them are of finite index, then all these homomorphisms have a Herbrand-Quotient and

$$\prod_{i=0}^{n} \mathrm{q}(h_i)^{(-1)^i} = 1.$$

(**Hint :** For the analogous concept see the concept of index in Supplement S8.9.)

(c) If $h:G \to F$ and $j:F \to E$ have Herbrand-Quotients, then the homomorphism $jh:G \to E$ also has Herbrand-Quotient and $q(jh) = \mathrm{q}(j)\,\mathrm{q}(h)$.

(d) If $h:G \to F$ has a Herbrand-Quotient and if $j:G \to F$ is a homomorphism with a finite image, then $h+j$ also a Herbrand-Quotient and $q(h+j) = \mathrm{q}(h)$.

S8.11 Let $V' \to V \to V''$ be a complex of K-vector space with the homology spaces H and X be another K-vector space. Then the homology spaces of the complexes

$$\mathrm{Hom}_K(V'',X) \longrightarrow \mathrm{Hom}_K(V,X) \longrightarrow \mathrm{Hom}_K(V',X) \qquad \text{and}$$

$$\mathrm{Hom}_K(X,V') \longrightarrow \mathrm{Hom}_K(X,V) \longrightarrow \mathrm{Hom}_K(X,V'')$$

are canonically isomorphic to $\mathrm{Hom}_K(H,X)$ and $\mathrm{Hom}_K(X,H)$, respectively, see Supplement S8.3. In particular, if $X \neq 0$, then it follows from the exactness of one of the both Hom-sequences, the exactness of the original sequence.