E0 219 Linear Algebra and Applications / August-December 2016

(ME, MSc. Ph. D. Programmes)

Download from : http://www.math.iisc.ernet.in/patil/courses/courses/Current Courses/...

Tel : +91-(0)80-2293 2239/(Maths Dept. 3212) **E-mails :** dppatil@csa.iisc.ernet.in / patil@math.iisc.ernet.in

Lectures : Monday and Wednesday ; 11:00–12:30 **Venue:** CSA, Lecture Hall (Room No. 117)

Corrections by : **Nikhil Gupta** (nikhil.gupta@csa.iisc.ernet.in ; **Lab No.: 303**) /
Vineet Nair (vineetn90@gmail.com ; **Lab No.: 303**) /
Rahul Gupta (rahul.gupta@csa.iisc.ernet.in ; **Lab No.: 224**) /
Sayantan Mukherjee (meghanamande@gmail.com ; **Lab No.: 253**) /
Palash Dey (palash@csa.iisc.ernet.in ; **Lab No.: 301 , 333 , 335**)

Midterms : 1-st Midterm : Saturday, September 17, 2016; 15:00 – 17:00 **2-nd Midterm :** Saturday, October 22, 2016; 15:00 – 17:00

Final Examination : December ??, 2016, 09:00 – -12:00

Evaluation Weightage : Assignments : 20% **Midterms (Two) :** 30% **Final Examination :** 50%

Range of Marks for Grades (Total 100 Marks)						
	Grade S	**Grade A**	**Grade B**	**Grade C**	**Grade D**	**Grade F**
Marks-Range	>90	76 — 90	61 — 75	46 — 60	35 — 45	<35

	Grade A$^+$	**Grade A**	**Grade B$^+$**	**Grade B**	**Grade C**	**Grade D**	**Grade F**
Marks-Range	>90	81 — 90	71 — 80	61 — 70	51 — 60	40 — 50	<40

Supplement 10

Determinants

Permutations , Determinant functions , Determinant of a linear operator , Orientations , Determinants and Volumes

To understand and appreciate the Supplements which are marked with the symbol † one may possibly require more mathematical maturity than one may have! These are steps towards applications to various other branches of mathematics, especially to analysis, number theory and Affine and Projective Geometry.

Participants may ignore these Supplements — altogether or in the first reading!!

S10.1 For $n \geq 3$, the symmetric group $\mathfrak{S}_n$ is not abelian and for $n \geq 4$, the alternating group $\mathfrak{A}_n$ is not abelian.

S10.2 (Inversions of a permutation) In the case $I = \{1, \ldots, n\}$ the signature of a permutation $\sigma \in \mathfrak{S}(I) = \mathfrak{S}_n$ can also be computed by counting the so-called inversions.

For $\sigma \in \mathfrak{S}_n$ a pair $(i, j) \in I \times I$ is called a inversion of σ if $i < j$, but $\sigma(i) > \sigma(j)$. The number of inversions of σ is denoted by $\mathrm{z}(\sigma)$. For example :

(1) The transposition $\langle i, j\rangle \in \mathfrak{S}_n$, $i < j$, has the inversions $(i, i+1), \ldots, (i, j)$; $(i+1, j), \ldots, (j-1, j)$ and hence $\mathrm{z}(\langle i, j\rangle) = 2(j-i) - 1$.

(2) In the permutation $\sigma := \binom{1\ \ 2\ \ldots\ n}{n\ n{-}1 \ldots 1} \in \mathfrak{S}_n$ all the pairs (i, j) with $1 \leq i < j \leq n$ inversions and hence $\mathrm{z}(\sigma) = \binom{n}{2}$.

(3) The permutation $\sigma := \binom{12345}{31524} \in \mathfrak{S}_5$ has the inversions $(1,2)$, $(1,4)$, $(3,4)$ and $(3,5)$ and hence $\mathrm{z}(\sigma) = 4$.

In general, for an arbitrary permutation $\sigma \in \mathfrak{S}_n$, $\mathrm{Sign}\,\sigma = (-1)^{\mathrm{z}(\sigma)}$. (**Proof :** Since by Example (1) above a transposition has an odd number of inversions, it is enough to prove that : For $\sigma, \tau \in \mathfrak{S}_n$, $(-1)^{\mathrm{z}(\sigma\tau)} = (-1)^{\mathrm{z}(\sigma)}(-1)^{\mathrm{z}(\tau)}$. For $\sigma \in \mathfrak{S}_n$, clearly $(-1)^{\mathrm{z}(\sigma)} = \prod_{1 \leq i < j \leq n} \mathrm{Sign}\,(\sigma(j) - \sigma(i))$. Therefore $(-1)^{\mathrm{z}(\sigma\tau)} = \prod_{1 \leq i < j \leq n} \mathrm{Sign}\,(\sigma(\tau(j)) - \sigma(\tau(i))) = (-1)^{\mathrm{z}(\tau)} \prod_{1 \leq r < s \leq n} \mathrm{Sign}\,(\sigma(s) - \sigma(r)) = (-1)^{\mathrm{z}(\tau)}(-1)^{\mathrm{z}(\sigma)}$. The second equality follows from the fact that exactly there are $\mathrm{z}(\tau)$ pairs $(\tau(i), \tau(j))$, $1 \leq i < j \leq n$ such that their components are interchanged and for this we need to consider the set of all pairs (r, s), $1 \leq r < s \leq n$.)

S10.3 For the following permutations σ find the canonical cycle decompositions, representations as the product of transpositions, the number of inversions, the signatures, the inverse permutation σ^{-1} and the orders (in the permutation group) :

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 3 & 2 & 9 & 10 & 8 & 12 & 4 & 6 & 1 & 11 & 7 & 5 \end{pmatrix} \in \mathfrak{S}_{12}$. Moreover, compute the power σ^{51}.

(**Ans:** $\operatorname{Sign}\sigma = 1$, $\operatorname{Ord}\sigma = 12$.)

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 7 & 12 & 1 & 10 & 8 & 2 & 11 & 4 & 6 & 5 & 3 & 9 \end{pmatrix} \in \mathfrak{S}_{12}$. Moreover, compute the power σ^{51}.

(**Ans:** $\operatorname{Sign}\sigma = ?$, $\operatorname{Ord}\sigma = ??$.)

(c) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 1 & 4 & 10 & 12 & 5 & 7 & 11 & 2 & 15 & 14 & 9 & 8 & 6 & 3 & 13 \end{pmatrix} \in \mathfrak{S}_{15}$.

(**Ans:** $\operatorname{Sign}\sigma = ?$, $\operatorname{Ord}\sigma = ??$.)

(d) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\ 15 & 8 & 17 & 4 & 7 & 14 & 20 & 19 & 18 & 13 & 10 & 6 & 11 & 5 & 3 & 12 & 1 & 9 & 2 & 16 \end{pmatrix} \in \mathfrak{S}_{20}$.
Moreover, compute the power σ^{100}. (**Ans:** $\operatorname{Sign}\sigma = 1$, $\operatorname{Ord}\sigma = 84$.)

(e) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\ 17 & 19 & 11 & 6 & 12 & 2 & 20 & 8 & 10 & 18 & 1 & 13 & 5 & 15 & 9 & 4 & 3 & \text{‘}4 & 16 & 7 \end{pmatrix} \in \mathfrak{S}_{20}$.
Moreover, compute the power σ^{100}. (**Ans:** $\operatorname{Sign}\sigma = 1$, $\operatorname{Ord}\sigma = 60$ and $\sigma^{100} = \langle 5, 12, 13\rangle$.)

S10.4 For a subset $J \subseteq \{1,\ldots,n\}$ with $J = \{j_1,\ldots,j_m\}$, $j_1 < \cdots < j_m$, let σ_J be the so-called shuffle-permutation:

$$\sigma_J = \begin{pmatrix} 1 & \ldots & m & m+1 & \ldots & n \\ j_1 & \ldots & j_m & i_1 & \ldots & i_{n-m} \end{pmatrix} \in \mathfrak{S}_n,$$

where the numbers $i_1 < \cdots < i_{n-m}$ are the elements of the complement J' of J in $\{1,\ldots,n\}$. Show that the number of inversions of σ_J is $\mathrm{z}(\sigma_J) = \sum_{\mu=1}^m (j_\mu - \mu) = \left(\sum_{\mu=1}^m j_\mu\right) - \binom{m+1}{2}$. In particular, $\operatorname{Sign}(\sigma_J) = (-1)^{\mathrm{z}(\sigma_J)}$. (**Hint:** See Supplement S10.2. — The set of inversions of σ_J is $\{(\mu,\nu) \mid \mu = 1,\ldots,m,\ \nu = m+1,\ldots,n \ \text{ and } \ j_\mu > i_\nu\}$. — **Remark:** In general, it is important and difficult to compute the order of the shuffle-permutations in the permutation group $\mathfrak{S}_n$. For computations of the order of shuffle-permutations and applications, see the article: [**D. P. Patil** and **U. Storch:** Group Actions and Elementary Number Theory. *J. Indian Inst. Sci.* **91** (2011), No. 1, 1-45.])

S10.5 Let I, J be two finite sets, $|I| = m$, $|J| = n$, and $\sigma \in \mathfrak{S}(I)$, $\tau \in \mathfrak{S}(J)$. Then compute the sign of the following permutations:

(a) $(x_1,x_2) \mapsto (x_2,x_1)$ of $I \times I$.

(b) $\sigma \uplus \tau \in \mathfrak{S}(I \uplus J)$ with $(\sigma \uplus \tau)|_I = \sigma$, $(\sigma \uplus \tau)|_J = \tau$.

(c) $\sigma \times \tau \in \mathfrak{S}(I \times J)$ with $(\sigma \times \tau)(x,y) = (\sigma(x), \tau(y))$. (**Hint:** The permutation in (a) has the sign $(-1)^{\binom{m}{2}}$ and $\operatorname{Sign}(\sigma \uplus \tau) = \operatorname{Sign}\sigma \cdot \operatorname{Sign}\tau$ and $\operatorname{Sign}(\sigma \times \tau) = (\operatorname{Sign}\sigma)^n \cdot (\operatorname{Sign}\tau)^m$.)

S10.6 Let I be a finite set, $|I| = m$ and $\mathfrak{P}_r(I)$ be the set of the r-subsets of I, $0 \le r \le m$. For $\sigma \in \mathfrak{S}(I)$, compute the sign of the permutation induced by σ: $\mathfrak{P}_r(\sigma): J \mapsto \sigma(J)$ of $\mathfrak{P}_r(I)$. (**Ans:** $\operatorname{Sign}\left(\mathfrak{P}_r(\sigma)\right) = (\operatorname{Sign}\sigma)^{\binom{m-2}{r-1}}$, where we put $\binom{m-2}{-1} := 0$ for all $m \in \mathbb{N}$. — **Proof:** Note that $\mathfrak{P}_r(\sigma\tau) = \mathfrak{P}_r(\sigma)\mathfrak{P}_r(\tau)$ for $\sigma, \tau \in \mathfrak{P}_r(I)$. Therefore, it is enough to prove this assertion for a transposition $\sigma = \langle a,b\rangle$. Since $\mathfrak{E}_0(I) = \{\emptyset\}$, we may assume that $r \ge 1$. If $J \in \mathfrak{P}_r(I)$ and if either both $a \notin J$, $b \notin J$, or both $a, b \in J$, then $\sigma(J) = J$. Further, σ interchanges the subsets $\{a\} \cup J'$ and $\{b\} \cup J'$, $J' \in \mathfrak{P}_{r-1}(I \smallsetminus \{a,b\})$. Now, since $|\mathfrak{P}_{r-1}(I \smallsetminus \{a,b\})| = \binom{m-2}{r-1}$, the assertion follows. •
— **Remark:** If $m \ge 2$, then by Supplement S10.5 (b), σ induces a permutation $\mathfrak{P}(\sigma)$ on $\mathfrak{P}(I) = \uplus_{r=0}^m \mathfrak{P}_r(I)$ and $(\operatorname{Sign}\sigma)^{2^{m-2}} = \prod_{r=0}^m \operatorname{Sign}(\mathfrak{P}_r(\sigma))$.)

S10.7 A subgroup of the permutation group $\mathfrak{S}_n$, $n \in \mathbb{N}^+$, which contain an odd permutation contains equal number of even and odd permutations. (**Hint:** Let $\sigma \in H$ be an odd permutation. The left translation $\lambda_\sigma : H \to H$, $\tau \mapsto \sigma\tau$ is bijective (with inverse $\lambda_{\sigma^{-1}}$) and maps even permutations in H onto odd permutations in H.)

S10.8 **(a)** A permutation $\sigma \in \mathfrak{S}_n$, $n \in \mathbb{N}^+$ which is of odd order is an even permutation.

(b) The square σ^2 of a permutation $\sigma \in \mathfrak{S}_n$, $n \in \mathbb{N}^+$, is an even permutation.

(**Hint :** If the order of σ is odd, then all cycles in the canonical decomposition of σ have also odd order, since the order of σ is the LCM of these orders. Therefore, all these cycles are of odd lengths and hence even permutations. Therefore, their product is also even. (b) follows from $\operatorname{Sign}\sigma^2 = (\operatorname{Sign}\sigma)^2 = 1$. — **Remark :** More generally : If $H \subseteq G$ is a subgroup of a group G of index 2, then $a^2 \in H$ for all $a \in G$. Note that (b)⇒(a) : If σ is an element of an odd order m in an arbitrary group G, then $\sigma = \sigma^{m+1} = \tau^2$ with $\tau := \sigma^{(m+1)/2}$.)

S10.9 Let $\sigma = \langle i_0, \ldots, i_{k-1}\rangle$ be a cycle of length $k \geq 2$. What is the inverse of σ ? For which $m \in \mathbb{Z}$, σ^m is a cycle of length k ?

S10.10 Let $\sigma \in \mathfrak{S}_n$ and $m \in \mathbb{Z}$. Every orbit of σ of length k decomposes into $\gcd(k,m)$ orbits of the length $k/\gcd(k,m)$ of σ^m.

S10.11 Let I be a finite set. The inverse σ^{-1} of a permutation $\sigma \in \mathfrak{S}(I)$ has the same orbits and same sign as those of σ.

S10.12 Let $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ be the canonical prime factorisation of $m \in \mathbb{N}^*$. Then the permutation group $\mathfrak{S}_n$ contain an element of order m if and only if $n \geq p_1^{\alpha_1} + \cdots + p_r^{\alpha_r}$. Give an element of biggest possible order in the group $\mathfrak{S}_5$. For which $n \in \mathbb{N}$ there exists an element of order 3000 (respectively 3001) in the group $\mathfrak{S}_n$?

†**S10.13** Let T be a set of transpositions in the group $\mathfrak{S}_n$, $n \geq 1$. We associate the *graph* [1] Γ_T to T as follows : the vertices of Γ_T are the numbers $1, \ldots, n$ and two vertices i and j with $i \neq j$ are joined by a edge if and only if the transposition $\langle i, j\rangle = \langle j, i\rangle$ belong to T. Let $\Gamma_1, \ldots, \Gamma_r$ be the connected components of Γ_T .

(a) The transpositions in T *generate the group* [2] $\mathfrak{S}_n$ if and only if Γ_T is connected, i.e. if any two vertices of Γ_T can be joined by the sequence of edges in Γ_T. The subgroup of $\mathfrak{S}_n$ generated by T is the product $\mathfrak{S}(\Gamma_1) \times \cdots \times \mathfrak{S}(\Gamma_r) \subseteq \mathfrak{S}_n$.

(b) If T is a generating system for the group $\mathfrak{S}_n$, then T has at least $n-1$ elements. (**Hint :** Let $\tau_1, \ldots, \tau_m$ be the elements of T (may be with repetitions) with $\tau_1 \cdots \tau_m = \mathrm{id}$. Then m is even and $m \geq 2\sum_{\rho=1}^{r}(|\Gamma_\rho| - 1)$.)

(c) Every generating system of $\mathfrak{S}_n$ consisting of transpositions contain a (minimal) generating system of $\mathfrak{S}_n$ with $n-1$ elements. (**Remarks :** The graphs corresponding to such a minimal generating systems are called *trees*. Every connected graph has a generating system which is a tree. See also remarks in Subsection 6.D. — There are exactly n^{n-2} generating systems consisting $n-1$ transpositions (*Cayley* [3]).

---

[1] **Simplicial Complexes and Graphs.** A *simplicial complex* $\mathcal{K}$ is a set $\mathbf{V}(\mathcal{K})$ called the *vertex set* (of $\mathcal{K}$) and a family of subsets of $\mathbf{V}(\mathcal{K})$, called *simplexes* (in $\mathcal{K}$) such that (i) for each $v \in \mathbf{V}(\mathcal{K})$, the singleton set $\{v\}$ is a simplex in K. and (ii) if $\mathbf{s}$ is a simplex in $\mathcal{K}$ then so is every subset of $\mathbf{s}$.

A simplex $\mathbf{s}$ in $\mathcal{K}$ is called a q-*simplex* if $\operatorname{card}(\mathbf{s}) = q+1$ and say that $\mathbf{s}$ has *dimension* q. For a simplicial complex $\mathcal{K}$, we put $\dim(\mathcal{K}) := \sup\{q \mid$ there exists a q-simplex in $\mathcal{K}\}$ and is called the *dimension* of $\mathcal{K}$. A simplicial complex of dimension ≤ 1 is called a *graph*.

An *edge* in $\mathcal{K}$ is an ordered pair (v_0, v_1) of vertices such that $\{v_0, v_1\}$ is a simplex in $\mathcal{K}$. If $\mathbf{e} = (v_0, v_1)$ is an edge in $\mathcal{K}$, then we put $v_0 = \alpha(\mathbf{e})$ and $v_1 = \varepsilon(\mathbf{e})$) and are called the *initial* and *end* points of $\mathbf{e}$, respectively.

A *path* γ in $\mathcal{K}$ of length n is a sequence $\mathbf{e}_1\mathbf{e}_2\cdots\mathbf{e}_n$ of edges in K with $\varepsilon(\mathbf{e}_i) = \alpha(\mathbf{e}_{i+1})$ for every $1 \leq i \leq n-1$. For a path $\gamma = \mathbf{e}_1\mathbf{e}_2\cdots\mathbf{e}_n$, we put $\alpha(\gamma) = \alpha(\mathbf{e}_1)$ and $\varepsilon(\gamma) := \varepsilon(\mathbf{e}_n)$ and are called the *initial* and *end* points of γ.

A simplicial complex $\mathcal{K}$ is called *connected* if for every pair (v_0, v_1) of vertices in $\mathcal{K}$ there exists a path α in $\mathcal{K}$ such that $\operatorname{orig}(\alpha) = v_0$ and $\operatorname{end}(\alpha) = v_1$.

[2] The smallest subgroup $\mathrm{H}(a_i \mid i \in I)$ of a group G containing the family a_i, $i \in I$, of elements in G, is called the *subgroup generated by the family* $a_i, i \in I$ (it is the intersection of the subgroups of G containing all $a_i, \in I$) and the family a_i, $i \in I$, is called a *generating system* for the subgroup $\mathrm{H}(a_i \mid i \in I)$. A family a_i, $i \in I$, is called a *generating system* for the group G if $G = \mathrm{H}(a_i \mid i \in I)$. We say that a group in *finitely generated* if there exists a finite family $a_1, \ldots, a_r \in G$ such that $G = \mathrm{H}(a_1, \ldots, a_r)$. Finite groups are clearly finitely generated. The groups $(\mathbb{Z}, +)$ and $(\mathbb{Z}_n, +_n)$ are generated by single elements, namely by 1 and $[1]_n$, respectively. Such groups are called *cyclic groups*. The groups $(\mathbb{Q}, +)$ and $(\mathbb{Q}^\times, \cdot)$ are not finite generated! (remember the Fundamental Theorem of Arithmetic)

[3] *Arthur Cayley* (1821-1895) an English mathematician and leader of the British school of pure mathematics that emerged in the 19th century. The most important of Cayley's work is in developing the algebra of matrices and work in non-euclidean and n-dimensional geometry.

For this prove somewhat general: For $1 \le k \le n$, let $f_{n,k}$ denote the number of forests with the vertex set $\{1,\ldots,n\}$ and exactly k *marked* trees (so-called r o o t - t r e e s), then $f_{n,n}=1$, $(n-k+1)f_{n,k-1}=n(k-1)f_{n,k}$ (by "grafting" one can get from a forest with $k \ge 2$ root-trees $n(k-1)$ forest with $k-1$ root-trees and by removing a edge at a time from a forest with $k-1$ root-trees $n-k+1$ forest with k root-trees) and hence $f_{n,k}=\binom{n-1}{k-1}n^{n-k}$, $1\le k\le n$. — The required number is $f_{n,1}/n$.)

(d) The transpositions $\langle 1,2\rangle$, $\langle 2,3\rangle,\ldots,\langle n-1,n\rangle$ (respectively $\langle 1,2\rangle$, $\langle 1,3\rangle,\ldots,\langle 1,n\rangle$) form a minimal generating system of $\mathfrak{S}_n$. (**Proof :** By induction on j, show that every transposition $\langle i,j\rangle$, $i<j$, is a product of transpositions of the form $\langle 1,2\rangle$, $\langle 2,3\rangle,\ldots,\langle n-1,n\rangle$. Induction starts at $j=i+1$ and for the inductive step, note that $\langle j,j+1\rangle\langle i,j\rangle\langle j,j+1\rangle=\langle i,j+1\rangle$. For the minimality, suppose that $\langle i,i+1\rangle$ can be dropped. Then, since for all other remaining transpositions the subsets $\{1,\ldots,i\}$ and $\{i+1,\ldots,n\}$ are invariant, every permutation $\sigma\in\mathfrak{S}_n$ with $\sigma(i)=i+1$, in particular, $\langle i,i+1\rangle$, can not represented as a product of the remaining transpositions. — For the second sequence of transpositions, every transposition $\langle i,j\rangle$, $i<j$ is a product $\langle 1,i\rangle\langle 1,j\rangle\langle 1,i\rangle=\langle i,j\rangle$. For minimality, suppose $\langle 1,\rangle i$ can be dropped. Then, since i is fixed under all other remaining transpositions, a permutation $\sigma\in\mathfrak{S}_n$ for which i is not fixed, in particular, $\langle 1,i\rangle$ can not be represented as a product of the remaining transpositions. •)

(e) An analogous assertion to the part (a) also hold for the alternating group. For a "triangle" $\triangle=\{a,b,c\}\in\mathfrak{P}_3(\{1,\ldots,n\})$, let $\alpha(\triangle)$ denote the set of the two 3-cycles $\langle a,b,c\rangle$, $\langle a,c,b\rangle=\langle a,b,c\rangle^{-1}$ (which is independent of an order or of "orientation" of the $\triangle$).

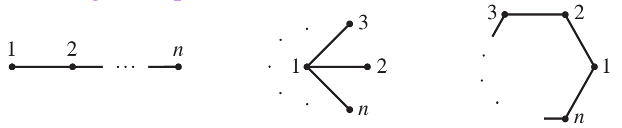
For 3-sets[4] $\triangle_1,\ldots,\triangle_m\in\mathfrak{P}_3(\{1,\ldots,n\})$, show that $\alpha(\triangle_1)\cup\cdots\cup\alpha(\triangle_m)$ generates the group $\mathfrak{A}(\Gamma_1)\times\cdots\times\mathfrak{A}(\Gamma_r)\subseteq\mathfrak{A}_n$, where $\Gamma_1,\ldots,\Gamma_r$ are the connected components of the graph with vertex-set $\{1,\ldots,n\}$ and whose edges belongs to any one of the triangle $\triangle_1,\ldots,\triangle_m$. (**Hint :** By induction on t prove that : If $\triangle_1,\ldots,\triangle_t$ are 3-sets with $\triangle_i\cap\triangle_{i+1}\neq\emptyset$ for $i=1,\ldots,t-1$, then $\alpha(\triangle_1)\cup\cdots\cup\alpha(\triangle_t)$ generates the alternating group $\mathfrak{A}(\triangle_1\cup\cdots\cup\triangle_t)$.)

Deduce that : The minimal number of 3-cycles which generates the group $\mathfrak{A}_n$, $n\ge 3$, is $\lceil (n-1)/2\rceil$. Give three 3-cycles which generates the group $\mathfrak{A}_5$, but no two (= $\lceil (5-1)/2\rceil$) among them generate the group $\mathfrak{A}_5$.(**Hint :** Check that $\langle 1,2,3\rangle$, $\langle 1,2,4\rangle$, $\langle 1,2,5\rangle$, is a minimal generating system for the group $\mathfrak{A}_5$.)

(f) For $n\ge 3$, the 3-cycles $\langle 1,2,3\rangle$, $\langle 2,3,4\rangle,\ldots,\langle n-2,n-1,n\rangle$ (resp. $\langle 1,2,3\rangle$, $\langle 1,2,4\rangle,\ldots,\langle 1,2,n\rangle$) form a generating system for the alternating group $\mathfrak{A}_n$. (**Hint :** Note that (e)⇒(f).)

(g) If n is even (resp. odd), then the cycles $\langle 1,2,3\rangle$, $\sigma:=\langle 1,2,3,\ldots,n\rangle$ (resp. $\langle 1,2,3\rangle$, $\tau:=\langle 2,3,\ldots,n\rangle$) generate the alternating group $\mathfrak{A}_n$. (**Hint :** Since $\sigma^k\langle 1,2,3\rangle\sigma^{-k}=\langle k+1,k+2,k+3\rangle$ and $\tau^k\langle 1,2,3\rangle\tau^{-k}=\langle 1,k+2,k+3\rangle$, $k=0,\ldots,n-3$, it follows that (e)⇒(g).)

†**S10.14** A permutation $\sigma\in\mathfrak{S}_n$ with s orbits has a representation as a product of $n-s$ transpositions and no representation as a product of less number of $n-s$ transpositions. (**Remark :** This exercise has a following natural generalisation : Let $T\subseteq\mathfrak{S}_n$ be a set of transpositions which generates the group $\mathfrak{S}_n$ (for example, by the given connected graph $\Gamma=\Gamma_T$ on the vertex set $\{1,\ldots,n\}$, see Supplement S10.12 (a)). For $\sigma\in\mathfrak{S}_n$ determine the minimum $\ell(\sigma)=\ell_T(\sigma)$ of the $m\in\mathbb{N}$, for which there is a representation $\sigma=\tau_1\cdots\tau_m$ with $\tau_i\in T$. Incidentally, $\ell(\sigma)=\ell(\sigma^{-1})$, and $d(\sigma_1,\sigma_2):=\ell(\sigma_2\sigma_1^{-1})$, $\sigma_1,\sigma_2\in\mathfrak{S}_n$, is a metric on $\mathfrak{S}_n$. Further, the left- and right-translations $\lambda_\tau:\mathfrak{S}_n\to\mathfrak{S}_n$, $\sigma\mapsto\tau\sigma$ and $\rho_\tau:\mathfrak{S}_n\to\mathfrak{S}_n$, $\sigma\mapsto\sigma\tau$ are distance preserving (for this, it enough to check that $d(\tau\sigma_1,\tau\sigma_2)=\ell(\tau\sigma_2\cdot(\tau\sigma_1)^{-1})\ell(\tau\sigma_2\sigma_1^{-1}\tau^{-1})=\ell(\sigma_2\sigma_1^{-1})=d(\sigma_1,\sigma_2)$ and similarly, $d(\sigma_1\tau,\sigma_2\tau)=d(\sigma_1,\sigma_2)$ for every *transposition* $\tau\in\mathfrak{S}_n$). For Γ_T, besides the complete graphs, one can also consider the following examples :



etc.

For the first of these graph see Exercise 10.2. For $T\subseteq T'$, it is clear that $\ell_{T'}\le\ell_T$.)

S10.15 **(a)** The cycles $\langle 1,2\rangle$, $\langle 2,\ldots,n\rangle$ generate the group $\mathfrak{S}_n$, $n\ge 2$. (**Proof :** Since ord $\langle 2,3,\ldots,n\rangle=n-1$, $\langle 2,3,\ldots,n\rangle^{n-1}=\mathrm{id}$ and $\langle 2,3,\ldots,n\rangle^{n-2}=\langle 2,3,\ldots,n\rangle^{-1}$. By Supplement S10.13 (d), it is enough to prove that every transposition of the form $\langle 1,j\rangle$ is a product of given cycles. This is proved by induction on j.

[4]For any $r\in\mathbb{N}$, let $\mathfrak{P}_r(I)$ denote the subset of the power set $\mathfrak{P}(I)$ of a set I consisting of subsets $J\subseteq I$ of cardinality exactly r. With this r-s e t is an element $\mathfrak{P}_r(\{1,\ldots,n\})$, i. e. a subset of $\{1,\ldots,n\}$ of cardinality r.

Induction begins at $j=2$ and the inductive step follows from $\langle 1, j+1\rangle = \langle 2,3,\ldots,n\rangle\langle 1,j\rangle\langle 2,3,\ldots,n\rangle^{-1} = \langle 2,3,\ldots,n\rangle\langle 1,j\rangle\langle 2,3,\ldots,n\rangle^{n-2}$.)

(b) The cycles $\langle 1,2\rangle$, $\langle 1,2,\ldots,n\rangle$ generate the group $\mathfrak{S}_n$, $n\geq 2$. More generally : if $k,n\in\mathbb{N}$ are natural numbers with $1<k\leq n$, then the cycles $\langle 1,k\rangle$, $\langle 1,2,\ldots,n\rangle$ generate the group $\mathfrak{S}_n$ if and only if $\gcd(k-1,n)=1$. In particular, the cycles $\langle 1,n\rangle$, $\langle 1,\ldots,n\rangle$ generate the group $\mathfrak{S}_n$, $n\geq 2$. (**Hint :** Use Supplement S10.12 (d).)

S10.16 (B o s s - P u z z l e) Let $r,s\in\mathbb{N}^*$, $r,s\geq 2$. In an right side box there are $rs-1$ numbers $1,2,\ldots,rs-1$ are arranged in a $r\times s$-rectangle (as shown in the left-rectangle which is made up of equal rs sliding square-blocks) by the permutation

$$\nu = \begin{pmatrix} 1 & 2 & 3 & \cdots & rs-2 & rs-1 \\ \nu_1 & \nu_2 & \nu_3 & \cdots & \nu_{rs-2} & \nu_{rs-1}\end{pmatrix} \in \mathfrak{S}_{rs-1}$$

ν_1	$\cdots$	ν_{s-1}	ν_s
ν_{s+1}	$\cdots$	ν_{2s-1}	ν_{2s}
$\vdots$	$\cdots$	$\vdots$	$\vdots$
$\nu_{(r-1)s+1}$	$\cdots$	ν_{rs-1}	#

1	$\cdots$	$s-1$	s
$s+1$	$\cdots$	$2s-1$	$2s$
$\vdots$	$\cdots$	$\vdots$	$\vdots$
$(r-1)(s-1)+1$	$\cdots$	$rs-1$	#

The lower-right corner square-block marked with # is kept free. The goal is to reposition the square-blocks by sliding the square-blocks (one at a time) into the standard-configuration (shown in left-hand table). Show that this possible if and only if the permutation $\nu\in\mathfrak{S}_{rs-1}$ is even.

(**Remark :** The special case $r=4$ and $s=4$ is the (original) 15-puzzle [5]:

5	2	3	10
15	6	9	8
14		4	12
13	1	7	11

	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

This puzzle has inspired a sizable number of articles and references in the mathematical literature. Most references explain the impossibility of obtaining odd permutations, but the result that every even permutation is indeed possible is proved by few authors and a number of them give unnecessarily complicated explanations. Indeed, H e r s t e i n and K a p l a n s k y in (see: [Herstein, I. N. and Kaplansky, K.: *Matters Mathematical*, Chelsea, New York, 1978, 114-115]) write that “no really easy proof seems to be known”. — **Hint :** A s i m p l e m o v e interchanges the blank-square # with adjacent to it; for example, there are two beginning simple moves, namely, either interchange # and ν_{rs-1} or interchange # and $\nu_{(r-1)s}$. To analyze the game, note that each simple move is a special kind of transposition, namely, one that moves #. Moreover, performing a simple move corresponding to a special transposition τ from a position corresponding to the permutation σ yields a new position (corresponding to the permutation $\tau\sigma$). For example, if ν is the position above and $\tau=\langle \#,\nu_{rs-1}\rangle$, then $\tau\nu(\#)=\tau(\#)=\nu_{rs-1}$, $\tau\nu(rs-1)=\tau(\nu_{rs-1})=\#$ and $\tau\nu(i)=i$ for all other i. Therefore to come to the standard position, one needs special transpositions $\tau_1,\tau_2,\ldots,\tau_m$ such

[5]The 15-puzzle (also called G e m P u z z l e, B o s s P u z z l e, G a m e o f F i f t e e n, M y s t i c S q u a r e and many others) was "invented" by N o y e s P a l m e r C h a p m a n, a postmaster in Canastota, New York as early as 1874. The game became a craze in the U. S. in February 1880, Canada in March, Europe in April, but that craze had pretty much dissipated by July.

S a m u e l L o y d (1841-1911) an American chess player-composer, puzzle author, and recreational mathematician, claimed from 1891 until his death in 1911 that he invented the 15-puzzle. This is false — Loyd had nothing to do with the invention or popularity of the puzzle. Later interest was fuelled by Loyd offering a \$1,000 prize for anyone who could provide a solution for achieving a particular combination specified by Loyd, namely reversing the 14 and 15, i. e. $\sigma=\langle 14,15\rangle$. This was impossible, as had been shown over a decade earlier by J o h n s o n and S t o r y (1879), (see: [Johnson, W. W.; Story, W. E.: Notes on the 15-Puzzle, *American Journal of Mathematics*, **2** (4), (1879), 397-404]) as it required an even permutation. R o b e r t J a m e s “ B o b b y ” F i s c h e r (1943-2008) an American chess Grandmaster and the 11-th World Chess Champion, was an expert at solving the 15-Puzzle and had demonstrated on Nov. 8, 1972 a solution within 25 seconds. Today the puzzle appears on some computer screen savers and a version is distributed with every Macintosh computer. For larger versions of the n-puzzle, finding a solution is easy, but the problem of finding the shortest solution is *NP-hard* (??).

that $\tau_m \cdots \tau_2 \tau_1 \nu = \mathrm{id}$. Each simple move takes # up, down , left or right. Therefore the total number m of moves is $u+d+\ell+r$, where u, d, ℓ, r are the numbers of up, down, left, right moves, respectively. If # is to return at the position where it was, then $u=d$ and $\ell=r$. Therefore the total number of moves must be $m=2u+2r$ even. The permutation $\nu \in \mathfrak{S}_{16}$ corresponding to the configuration in the above picture is $\nu = \langle 1,15,14,13,3,2\rangle\langle 4,12,11,5\rangle\langle 6,10\rangle\langle 7,9,8\rangle$ is an odd permutation and hence it is not possible to bring it to the standard configuration. For the converse, use Supplement S10.13 (f) to reduce the problem to the cases $s=2$, $r=2$ or 3. —The permutations for which this is possible form a subgroup of $\mathfrak{S}_n$, in fact, it is the alternating group $\mathfrak{A}_n$ on n symbols.

—How to solve the 15-Puzzle for the magic square painted in the Dürers painting (where the number 16 represents the empty square, see the right picture above)?

—How can one convert the sequence of alphabets on the left side into the quotation of J. Sylvester (1814-1897) given on the right side. (see also a book by J. DieudonnÃl' (1906-1992)).

a	a	a	c	c	e	e	e	f	h	h	i
i	M	M	m	n	o	o	R	s	s	s	t
			t	t	u	–					

M	a	t	h	e	m	a	t	i	c	s	–
M	u	s	i	c		o	f		t	h	e
			R	e	a	s	o	n			

—For more such problems of this kind see : [Wilson, R.M.: Graph Puzzles, Homotopy, and the Alternating Group, Journal of Combinatorial Theory (B) **16**, 86-96 (1974).])

S10.17 Let $n \in \mathbb{N}^+$. Show that

(a) The number of permutations $\tau \in \mathfrak{S}_n$ which commute with the permutation $\sigma \in \mathfrak{S}_n$ of the type $(\nu_1,\ldots,\nu_n)$ is $\nu_1!\cdots\nu_n!\,1^{\nu_1}\cdots n^{\nu_n}$. (**Hint :** These permutations form the centraliser $\mathrm{C}_{\mathfrak{S}_n}(\sigma)$ of σ, see Example 9.A.20.)

(b) The number of involutions, , i. e., $\sigma^2 = \mathrm{id}$ (called reflection) in $\mathfrak{S}_{2n}$ without any fixed point in $\mathfrak{S}_{2n}$ is $1\cdot 3\cdots(2n-1) = (2n)!/n!\,2^n$ $(\sim \sqrt{2}\,(2n/e)^n$ for $n\to\infty)$.

(c) The number of involutions (reflections) in $\mathfrak{S}_n$ is $\sum_{k\geq 0}\binom{n}{2k}\dfrac{(2k)!}{k!\,2^k}$.

(d) The number of permutations in $\mathfrak{S}_n$ with exactly t orbits is the Stirling's number of first kind $s(n,t)$. (—The Stirling's numbers $s(m,n)$, $0\leq n\leq m$, of first kind are defined by the equation: $\displaystyle\binom{x}{m} = \frac{1}{m!}\sum_{n=0}^{m}(-1)^{m-n}s(m,n)\,x^n$ (and otherwise $s(m,n)=0$). They clearly satisfy the recursion-formula : $s(0,n)=\delta_{0n}$ and $s(m+1,n) = m\,s(m,n)+s(m,n-1)$.)

(e) The number of permutations in $\mathfrak{S}_n$ such that its canonical decomposition contain a (and hence exactly one) cycle of length $>n/2$, is $n!\big(\sum_{n/2<k\leq n}1/k\big)$ $(\sim n!\ln 2$ for $n\to\infty)$. (**Proof :** Let $1<k\leq n$. A cycle $\langle i_0,\ldots,i_{k-1}\rangle$ of length k in $\mathfrak{S}_n$ is determined by the injective map $\{0,\ldots,k-1\}\to\{1,\ldots,n\}$, $\nu\mapsto i_\nu$, where two such injective maps σ_1 and σ_2 define the same cycle if and only if $\sigma_1=\sigma_2\,\varphi$ with an element φ in the cyclic subgroup of $\mathfrak{S}(\{0,\ldots,k-1\})$ generated by the cycle $\langle 0,\ldots,k-1\rangle$. Therefore, there are $[n]_k/k = n!/k\cdot(n-k)!$ cycles of length k in $\mathfrak{S}_n$. Since a permutation in $\mathfrak{S}_n$ has at most one cycle of the length $k>n/2$, for such a cycle there are exactly $(n-k)!$ permutations such that this cycle occurs in its canonical decomposition. Therefore altogether, there are $\displaystyle\sum_{n/2<k\leq n}(n-k)!\cdot\frac{n!}{k\cdot(n-k)!} = n!\sum_{n/2<k\leq n}1/k = n!\,(H_n-H_{n/2})$ permutations in $\mathfrak{S}_n$ such that a cycle of length $>n/2$ occur in their canonical decomposition ($H_x = \displaystyle\sum_{k\in\mathbb{N}^*,\,k\leq x}1/k$ for $x\in\mathbb{R}_+^\times$ are the harmonic numbers.) The asymptotic representation $\displaystyle\sum_{n/2<k\leq n}1/k\sim\ln 2$ for $n\to\infty$ follows directly from $\displaystyle\sum_{n/2<k\leq n}1/k = \sum_{1\leq k\leq n}(-1)^{k-1}/k$ and $\displaystyle\sum_{k=1}^{\infty}(-1)^{k-1}/k = \ln 2$, or also from $H_x = \ln x+\gamma+O(1/x)$ for $x\to\infty$. •

—**Remark :** *The probability that a permutation in $\mathfrak{S}_n$ has a cycle of length $>n/2$ in its canonical decomposition is $H_n-H_{n/2}$ and for $n\to\infty$ symptotically equal to* $\ln 2 = 0.693\ldots$. – For an application, see Exercise 10.2.)

(f) The number of permutations in $\mathfrak{S}_n$ without any fixed point is $n!\big(\sum_{k=0}^{n}(-1)^k/k!\big)$ $(\sim n!/e$ for $n\to\infty)$. (**Hint :** For counting use the Inclusion Exclusion Principle.)

S10.18 **(a)** Using the simplicity of the alternating group $\mathfrak{A}_n$, $n\geq 5$, prove that the group $\mathfrak{A}_n$ is the only non-trivial normal subgroup of the group $\mathfrak{S}_n$ for $n \geq 5$. (**Hint :** See Example 9.A.23.)

(b) Let $n \geq 2$ be a natural number. Show that the group $\mathfrak{S}_n$ is isomorphic to a subgroup of $\mathfrak{A}_{n+2}$, but not isomorphic to any subgroup of $\mathfrak{A}_{n+1}$.

S10.19 **(a)** The groups $\mathfrak{A}_4$ and $\mathfrak{V}_4$ are the only non-trivial normal subgroups in $\mathfrak{S}_4$.

(b) The group $\mathfrak{V}_4$ is the only non-trivial normal subgroup in $\mathfrak{A}_4$. (**Hint :** See Example 9.A.23.)

S10.20 **(a)** For a natural number $n \geq 2$, $\mathrm{Sign} : \mathfrak{S}_n \to \{-1, 1\}$ is the only non-trivial group homomorphism. (**Hint :** $\langle ab \rangle$ and $\langle cd \rangle$ be two transpositions $\mathfrak{S}_n$. If $\sigma \in \mathfrak{S}_n$ be an arbitrary permutation with $a \mapsto c,\, b \mapsto d$, then $\sigma\langle ab \rangle\sigma^{-1} = \langle cd \rangle$ and so every homomorphism $\varphi : \mathfrak{S}_n \to \{1, -1\}$ have the same value on all transpositions. If this value is 1, then φ; if it is -1, then $\varphi = \mathrm{Sign}$.)

(b) Show that $\mathfrak{A}_n = [\mathfrak{S}_n, \mathfrak{S}_n]$ ($=$ the commutator subgroup [6] of $\mathfrak{S}_n$).

S10.21 Let I be a finite set and let $\sigma \in \mathfrak{S}(I)$ be a permutation of I. If the order $\mathrm{Ord}\,\sigma = p^m$ is a prim power, then $n := |I| \equiv |\mathrm{Fix}\,\sigma| \,(\mathrm{mod}\, p)$, where $\mathrm{Fix}\,\sigma := \{a \in I \mid \sigma(a) = a\}$ is the fixed point set of σ. In particular, : (1) If n is not divisible by p, then σ has at least one fixed point. (2) If n is divisible by p, then the number of fixed points of σ is also divisible by p. (**Remark :** This is a special case of the assertion at the end of Example 6.E.5.)

S10.22 Which of the following maps $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ are bilinear, symmetric resp. alternating?

(a) $f\big((x_1,x_2),(y_1,y_2)\big) := x_1 + y_2$. **(b)** $f\big((x_1,x_2),(y_1,y_2)\big) := x_1 y_2$.

(c) $f\big((x_1,x_2),(y_1,y_2)\big) := x_1 x_2 - y_1 y_2$. **(d)** $f\big((x_1,x_2),(y_1,y_2)\big) := x_1 y_2 - y_1 x_2$.

(e) $f\big((x_1,x_2),(y_1,y_2)\big) := x_1 y_2 + y_1 x_2$.

S10.23 Let V and W be K-vector spaces, I be a finite indexed set and $f : V^I \to W$ be a multi-linear map. Let $g : U \to V$ and $h : W \to X$ be K-vector space homomorphisms. Then $h \circ f \circ g^I : U^I \to X$ is a multi-linear map, where g^I is defined by $g^I\big((u_i)\big) := \big(g(u_i)\big)$, $(u_i) \in U^I$. If f is symmetric (respectively skew-symmetric, alternating), then so is $h \circ f \circ g^I$.

S10.24 Let v_j, $j \in J$ be a basis of the K-vector space V and let $w_{(j_i)}$, $(j_i) \in J^I$ be a family of elements of the K-vector space W, where I is a finite indexed set. Then there exists a unique K-multi-linear map $f : V^I \to W$ such that $f\big((v_{j_i})_{i \in I}\big) = w_{(j_i)}$, $(j_i) \in J^I$. If V and W are finite dimensional, then the K-vector space of the multi-linear maps from V^I into W has the dimension $(\mathrm{Dim}_K V)^{|I|} \cdot \mathrm{Dim}_K W$.

S10.25 A n-linear map $f : V^n \to W$ of K-vector spaces is alternating if $f(x_1, \ldots, x_n) = 0$ for every n-tuple $(x_1, \ldots, x_n)$ in which two *consecutive* components are equal. (**Proof :** By induction on $d > 0$, we shall show that $f(x_1, \ldots, x_n) = 0$ for all $i, j \in \{1, \ldots, n\}$ with $|i - j| = d$, if in the n-tuple $(x_1, \ldots, x_n)$ the i-th and the j-th components are equal. The case $d = 1$ is the hypothesis and so induction starts. For the inductive step we choose a $k \in \{1, \ldots, n\}$ in between i and j. Then $|i - k|$ and $|j - k|$ are smaller than d, and hence by the induction hypothesis

$$\begin{aligned} 0 = f(\ldots, x+y, \ldots, x+y, \ldots, x, \ldots) &= f(\ldots, x, \ldots, x, \ldots, x, \ldots) + f(\ldots, y, \ldots, x, \ldots, x, \ldots) \\ &+ f(\ldots, x, \ldots, y, \ldots, x, \ldots) + f(\ldots, y, \ldots, y, \ldots, x, \ldots) = f(\ldots, x, \ldots, y, \ldots, x, \ldots), \end{aligned}$$

where only the i-th, k-th and j-th components in the arguments are noted, the remaining are not altered.)

S10.26 Let K be a field and let V, W be vector spaces over K. Let $f : V^n \to K$ be an alternating multi-linear form on V and let $g : V \to W$ be a K-linear map. Show that the map

$$(x_0, \ldots, x_n) \longmapsto \sum_{i=0}^{n} (-1)^i f(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)\, g(x_i)$$

---

[6] For an arbitrary group G, the subgroup generated (see Footnote 8) by the commutators $[a,b] := aba^{-1}b^{-1}$, $a, b \in G$, is called the commutator subgroup or the derived group of G; it is usually denoted by $[G,G]$ or by $\mathrm{D}(G)$. Clearly, G is abelian if and only if $[G,G]$ is trivial. More generally, $[G,G]$ is a normal subgroup of G and the quotient group $G/[G,G]$ is abelian.

is an alternating K-multi-linear map $V^{n+1} \to W$. (**Proof:** The map is obviously multi-linear. By Supplement S10.25 it is enough to show that it vanish on every $(n+1)$-tuple with two equal consecutive components, say $x_i = x_{i+1} =: x$. Since f is alternating, in the above sum all terms except the i-th and the $(i+1)$-th term, are all 0. The remaining sum of two terms is:

$$(-1)^i f(x_0,\ldots,x_{i-1},x_{i+1},x_{i+2},\ldots,x_n)\,g(x_i) + (-1)^{i+1} f(x_0,\ldots,x_{i-1},x_i,x_{i+2},\ldots,x_n)\,g(x_{i+1})$$
$$= (-1)^i \big(f(x_0,\ldots,x_{i-1},x,x_{i+2},\ldots,x_n)\,g(x) - f(x_0,\ldots,x_{i-1},x,x_{i+2},\ldots,x_n)\,g(x)\big) = 0.)$$

S10.27 Let A be a K-vector space of dimension n with a $(n+1)$-multi-linear map $A^{n+1} \to A$, $(x_0,\ldots,x_n) \mapsto x_0 \cdots x_{n+1}$. Then show that $\sum_{\sigma\in\mathfrak{S}_{n+1}} (\mathrm{Sign}\,\sigma)\, x_{\sigma 0}\cdots x_{\sigma n} = 0$ for all $x_0,\ldots,x_n \in A$. (**Hint:** By Theorem 9.B.7 the map $(x_0,\ldots,x_n) \mapsto \sum_{\sigma\in\mathfrak{S}_{n+1}} (\mathrm{Sign}\,\sigma)\, x_{\sigma 0}\cdots x_{\sigma n}$ is alternating $(n+1)$-linear map and by Corollary 9.B.6 it is 0, since $\mathrm{Dim}\,A = n$. – We mention the following example: Let $A \times A \to A$ be a K-bilinear (or an arbitrary) operation $(x,y) \mapsto xy$ on A. Then $\sum_{\sigma\in\mathfrak{S}_{n+1}} (\mathrm{Sign}\,\sigma)\, x_{\sigma 0}\cdots x_{\sigma n} = 0$ for all $x_0,\ldots,x_n \in A$, if we compute all the $(n+1)$-fold products with one and the same fixed given rule of parentheses. — There are $\frac{1}{n+1}\binom{2n}{n}$ possible rules of parentheses.)

S10.28 For the matrices

$$\mathfrak{A} := \begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathfrak{B} := \begin{pmatrix} 5 & 5 & 3 & 1 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 3 & 1 & 1 & 2 \end{pmatrix}$$

compute the adjoint matrices, the determinants and the product $\mathfrak{A}\cdot\mathrm{Adj}\,\mathfrak{A}$ and $\mathfrak{B}\cdot\mathrm{Adj}\,\mathfrak{B}$.

S10.29 Determine for which $a \in \mathbb{R}$ the following systems of linear equations over $\mathbb{R}$ has exactly one solution and in this case find the solution by the Cramer's rule:

$$(\mathbf{1}) \quad \begin{aligned} ax_1 + x_2 + x_3 &= b_1 \\ x_1 + ax_2 + x_3 &= b_2 \\ x_1 + x_2 + ax_3 &= b_3\,. \end{aligned} \qquad\qquad (\mathbf{2}) \quad \begin{aligned} x_1 + x_2 - x_2 &= b_1 \\ 2x_1 + 3x_2 + ax_2 &= b_2 \\ x_1 + ax_2 + 3x_2 &= b_3\,. \end{aligned}$$

(**Answers:** (1) This system of equations has a unique solution if and only if $a \notin \{1,-2\}$ with the solution:

$$x_1 = \frac{b_1(a+1) - b_2 - b_3}{(a-1)(a+2)}\,,\quad x_2 = \frac{b_1(a+1) - b_1 - b_3}{(a-1)(a+2)}\,,\quad x_3 = \frac{b_1(a+1) - b_1 - b_2}{(a-1)(a+2)}$$

(2) This system of equations has a unique solution if and only if $a \notin \{2,-3\}$ with the solution:

$$x_1 = \frac{b_1(a-3) + b_2 - b_3}{a-2}\,,\quad x_2 = \frac{b_1(6-a) - 4b_2 + b_3(a+2)}{(a-2)(a+3)}\,,\quad x_3 = \frac{b_1(3-2a) + b_2(a-1) - b_3}{(a-2)(a+3)}\,.)$$

S10.30 Let $\mathfrak{A} = (a_{ij})$ be an $n\times n$-matrix over the field K. For $c_1,\ldots,c_n \in K^\times$, show that: $\mathrm{Det}\,(a_{ij}) = \mathrm{Det}\,\big(c_i c_j^{-1} a_{ij}\big)$. In particular, $\mathrm{Det}\,(a_{ij}) = \mathrm{Det}\,\big((-1)^{i+j} a_{ij}\big)$.

S10.31 Let $\mathfrak{A}$ and $\mathfrak{B}$ be $n\times n$ invertible matrices over the field K. Then show that:

(a) $\mathrm{Adj}\,(\mathfrak{A}\mathfrak{B}) = \mathrm{Adj}\,\mathfrak{B}\cdot\mathrm{Adj}\,\mathfrak{A}$. **(b)** $\mathrm{Adj}\,\mathfrak{A}^{-1} = (\mathrm{Adj}\,\mathfrak{A})^{-1}$.

(c) $\mathrm{Det}\,(\mathrm{Adj}\,\mathfrak{A}) = (\mathrm{Det}\,\mathfrak{A})^{n-1}$. **(d)** $\mathrm{Adj}\,(\mathrm{Adj}\,\mathfrak{A}) = (\mathrm{Det}\,\mathfrak{A})^{n-2}\,\mathfrak{A}$.

(**Remark :** All these formulas, except (b) are also valid for not-invertible matrices ; for (d) assume $n > 1$.)

S10.32 Let $\mathfrak{A}$ be a non-invertible $n\times n$-matrix over the field K, $n \geq 1$. Show that the rank of the adjoint matrix $\mathrm{Adj}\,\mathfrak{A}$ is:

$$\mathrm{Rank}\ \mathrm{Adj}\,\mathfrak{A} = \begin{cases} 1, & \text{if } \mathrm{Rank}\,\mathfrak{A} = n-1\,, \\ 0, & \text{if } \mathrm{Rank}\,\mathfrak{A} < n-1\,, \end{cases}$$

Moreover, if $\mathrm{Rank}\,\mathfrak{A} = n-1$, then show that every non-zero column of $\mathrm{Adj}\,\mathfrak{A}$ generates the kernel of $\mathfrak{A}$, i. e. the space of all $\mathfrak{x} \in K^n$ with $\mathfrak{A}\mathfrak{x} = 0$.

S10.33 The $n\times n$-matrix $\mathfrak{A}' = (a'_{ij})$ obtained from the $n\times n$-matrix $\mathfrak{A} = (a_{ij})$ by reflection through the anti-diagonal, i. e., $a'_{ij} = a_{n-j+1,n-i+1}$. Then show that $\operatorname{Det}\mathfrak{A}' = \operatorname{Det}\mathfrak{A}$, i. e.,

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2,n-1} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n1} & a_{n2} & \cdots & a_{n,n-1} & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{nn} & a_{n-1,n} & \cdots & a_{2n} & a_{1n} \\ a_{n,n-1} & a_{n-1,n-1} & \cdots & a_{1,n-1} & a_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n2} & a_{n-1,2} & \cdots & a_{22} & a_{12} \\ a_{n1} & a_{n-1,1} & \cdots & a_{21} & a_{11} \end{vmatrix}.$$

(**Hint:** Use $\operatorname{Det}\mathfrak{A} = \operatorname{Det}{}^{\mathrm{t}}\mathfrak{A}$, see Theorem 9.D.1 and the permutation $\sigma : \{1,\ldots,n\} \to \{1,\ldots,n\}$, $i \mapsto n-i+1$ on the rows or columns of ${}^{\mathrm{t}}\mathfrak{A}$ and use Rule (3) before Theorem 9.D.2 to conclude: $\operatorname{Det}{}^{\mathrm{t}}\mathfrak{A}' = \operatorname{Det}\left(a'_{ij}\right) = \operatorname{Det}\left(a_{n-j+1,n-i+1}\right) = \operatorname{Det}\left(a_{n-j+1,\sigma(i)}\right) = \operatorname{Sign}(\sigma)\operatorname{Det}\left(a_{\sigma(j),i}\right) = \operatorname{Sign}(\sigma)\operatorname{Sign}(\sigma)\operatorname{Det}\left(a_{j,i}\right) = (\operatorname{Sign}(\sigma))^2\operatorname{Det}{}^{\mathrm{t}}\mathfrak{A} = \operatorname{Det}\mathfrak{A}$.)

S10.34 Let $x_1,\ldots,x_n \in K^n$ be columns of the matrix $\mathfrak{A} \in \mathrm{M}_n(K)$.

(a) Let $I, J \subseteq \{1,\ldots,n\}$ be $(n-r)$-element subsets with the complements $I' = \{i_1,\ldots,i_r\}$, $J' = \{j_1,\ldots,j_r\}$, $1 \le i_1 < \cdots < i_r \le n$, $1 \le j_1 < \cdots < j_r \le n$. In the matrix $\mathfrak{A}$ replace the columns with numbers $j_1,\ldots,j_r$ by the standard basis vectors $e_{i_1},\ldots,e_{i_r}$, then the determinant of this matrix is the higher cofactor $(-1)^{\sum_{\rho=1}^{r}(i_\rho+j_\rho)}\operatorname{Det}\mathfrak{A}_{I,J}$, where the matrix $\mathfrak{A}_{I,J}$ is obtained from the matrix $\mathfrak{A}$ by removing the rows and columns with numbers $i_1,\ldots,i_r$ and $j_1,\ldots,j_r$, resp.

(Note that the usual cofactor $(-1)^{i+j}A_{ij}$ correspond to the $(n-1)$-element subsets $I = \{1,\ldots,\hat{\imath},\ldots,n\}$ and $J = \{1,\ldots,\hat{\jmath},\ldots,n\}$. — **Proof:** Interchanging the rows with numbers $i_1,\ldots,i_r$ in altogether $\sum_{\rho=1}^{r}(i_\rho-\rho)$ steps bring to the positions $1,\ldots,r$ and interchanging the columns with numbers $j_1,\ldots,j_r$ in altogether $\sum_{\rho=1}^{r}(j_\rho-\rho)$ steps bring to the positions $1,\ldots,r$, we obtain a block matrix of the form $\begin{pmatrix} \mathfrak{E}_r & \mathfrak{A}' \\ 0 & \mathfrak{A}_{I,J} \end{pmatrix}$ with the determinant $\operatorname{Det}\mathfrak{A}_{I,J}$. •)

(b) Let $\mathfrak{B}$ be another $n\times n$-matrices with columns $y_1,\ldots,y_n \in K^n$. For a subset $J \subseteq \{1,\ldots,n\}$, let $\mathfrak{C}_J$ be the $n\times n$-matrix with the columns $z_1^{(J)},\ldots,z_n^{(J)}$, where

$$z_i^{(J)} := \begin{cases} x_i\,, & \text{if } i \in J\,, \\ y_i\,, & \text{if } i \notin J\,. \end{cases}$$

Show that

$$\operatorname{Det}(\mathfrak{A}+\mathfrak{B}) = \sum_{J\subseteq\{1,\ldots,n\}} \operatorname{Det}\mathfrak{C}_J\,.$$

(**Hint:** $\operatorname{Det}(\mathfrak{A}+\mathfrak{B}) = \Delta_{\mathfrak{e}}(x_1+y_1,\ldots,x_n+y_n) = \sum_{J\subseteq\{1,\ldots,n\}} \Delta_{\mathfrak{e}}(z_1^{(J)},\ldots,z_n^{(J)}) = \sum_{J\subseteq\{1,\ldots,n\}} \operatorname{Det}\mathfrak{C}_J$. — **Remark:** If $\mathfrak{B} = \operatorname{Diag}(b_1,\ldots,b_n)$ is a diagonal matrix, then $\operatorname{Det}\mathfrak{C}_J = b^{J'}\operatorname{Det}\mathfrak{A}_{J,J}$, where $b^I = \prod_{i\in I} b_i$ for $I \subseteq \{1,\ldots,\}$ and J' is the complement J. Altogether, we have:

$$\operatorname{Det}(\mathfrak{A}+\operatorname{Diag}(b_1,\ldots,b_n)) = \sum_{J\subseteq\{1,\ldots,n\}} b^{J'}\operatorname{Det}\mathfrak{A}_{J,J}\,.)$$

S10.35 **(a)** Suppose that a column (or a row) of the $n\times n$-matrix $\mathfrak{A}$ has all entries 1. For the cofactors $(-1)^{i+j}A_{ij}$, $i,j = 1,\ldots,n$, of $\mathfrak{A}$, show that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(-1)^{i+j}A_{ij} = \operatorname{Det}\mathfrak{A}\,.$$

(b) Let $\mathfrak{A} = (a_{ij})$ be an $n\times n$-matrix over the field K with the cofactors $(-1)^{i+j}A_{ij}$, $i,j = 1,\ldots,n$. Further, let

$$\mathfrak{I} := \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \in \mathrm{M}_n(K)$$

is the matrix with all the coefficients are equal to 1. Show that

$$\operatorname{Det}(\mathfrak{A}+a\,\mathfrak{I}) = \operatorname{Det}\mathfrak{A} + a\sum_{i=1}^{n}\sum_{j=1}^{n}(-1)^{i+j}A_{ij}\,.$$

(**Hint :** To apply Supplement S10.34 (b) with $\mathfrak{B} = a\mathfrak{J}$ and with the introduced matrices $\mathfrak{C}_J$. If $|J| \leq n-2$, then two distinct columns of $\mathfrak{C}_J$ are equal to ${}^{\rm t}(a,\ldots,a)$ and hence $\mathrm{Det}\,\mathfrak{C}_J = 0$. If $J = \{1,\ldots,j-1,j+1,\ldots,n\}$, then $\mathfrak{C}_J$ have same columns as $\mathfrak{A}$ except the j-th column which has all entries a. Expanding the determinant with respect to the j-th column, we get $\mathrm{Det}\,\mathfrak{C}_J = \sum_{i=1}^{n}(-1)^{i+j}aA_{ij}$. Finally, $\mathfrak{C}_J = \mathfrak{A}$ for $J = \{1,\ldots,n\}$. Therefore, by Supplement S10.34 (b), $\mathrm{Det}\,(\mathfrak{A}+a\mathfrak{J}_n) = \sum_{J\subseteq\{1,\ldots,n\}} \mathrm{Det}\,\mathfrak{C}_J = \mathrm{Det}\,\mathfrak{A} + \sum_{j=1}^{n}\sum_{i=1}^{n}(-1)^{i+j}aA_{ij}$. — **Remark :** Using the Remark in Supplement S10.34, it follows that

$$\mathrm{Det}\,(a\mathfrak{J}_n + \mathrm{Diag}(b_1,\ldots,b_n)) = b_1\cdots b_n + a\sum_{j=1}^{n} b_1\cdots\hat{b}_j\cdots b_n\,.)$$

S10.36 Let K be a field and $\mathfrak{A} = \big(a_{ij}\big) \in \mathrm{M}_n(K)$, $n \in \mathbb{N}^*$ be a matrix of rank ≤ 1. Show that :

$$\mathrm{Det}\,(a\mathfrak{E}+\mathfrak{A}) = a^n + a^{n-1}\sum_{i=1}^{n} a_{ii} \quad \text{for all} \quad a \in K\,.$$

S10.37 Let $\mathfrak{A} = (a_{ij}) \in \mathrm{M}_n(\mathbb{Q})$ be an invertible matrix with integer coefficients a_{ij}. Show that the coefficients of the inverse matrix $\mathfrak{A}^{-1}$ are again integers if and only if $\mathrm{Det}\,\mathfrak{A} = \pm 1$.

(**Hint :** If $\mathfrak{B}\in\mathrm{M}_m(\mathbb{Z})$, $m\in\mathbb{N}$, then $\mathrm{Det}\,\mathfrak{B}\in\mathbb{Z}$. Therefore, if $\mathfrak{A}, \mathfrak{A}^{-1}\in\mathrm{M}_n(\mathbb{Z})$, then from $(\mathrm{Det}\,\mathfrak{A})(\mathrm{Det}\,\mathfrak{A}^{-1}) = \mathrm{Det}\,(\mathfrak{A}\mathfrak{A}^{-1}) = \mathrm{Det}\,\mathfrak{E}_n = 1$, it follows that $\mathrm{Det}\,\mathfrak{A} = \mathrm{Det}\,\mathfrak{A}^{-1} \in \{\pm 1\}$. Conversely, if $\mathfrak{A}\in\mathrm{M}_n(\mathbb{Z})$ and $\mathrm{Det}\,\mathfrak{A} = \pm 1$, then $\mathfrak{A}^{-1} = (\mathrm{Det}\,\mathfrak{A})^{-1}\,\mathrm{Adj}\,\mathfrak{A} = \pm\mathrm{Adj}\,\mathfrak{A} \in \mathrm{M}_n(\mathbb{Z})$, since $\mathfrak{A}$ and also $\mathrm{Adj}\,\mathfrak{A}\in\mathrm{M}_n(\mathbb{Z})$.)

S10.38 Let $\mathfrak{A} \in \mathrm{M}_n(K)$ be an upper-triangular matrix. Then show that $\mathrm{Adj}\,\mathfrak{A}$ and $\mathfrak{A}^{-1}$ (if $\mathfrak{A}$ is invertible) are also upper-triangular matrices.

S10.39 Let f_{ij}, $i,j = 1,\ldots,n$ be differentiable functions on $D \subseteq \mathbb{K}$. Then show that

$$\begin{vmatrix} f_{11} & \cdots & f_{1n} \\ f_{21} & \cdots & f_{2n} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nn} \end{vmatrix}' = \begin{vmatrix} f'_{11} & \cdots & f'_{1n} \\ f_{21} & \cdots & f_{2n} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nn} \end{vmatrix} + \begin{vmatrix} f_{11} & \cdots & f_{1n} \\ f'_{21} & \cdots & f'_{2n} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} f_{11} & \cdots & f_{1n} \\ f_{21} & \cdots & f_{2n} \\ \vdots & \ddots & \vdots \\ f'_{n1} & \cdots & f'_{nn} \end{vmatrix}.$$

S10.40 If $\sigma \in \mathfrak{S}(I)$ is a permutation of the finite indexed I and let

$$\mathfrak{P}_\sigma = (\delta_{i\sigma(j)}) \in \mathrm{M}_I(K)$$

be the *permutation matrix associated to* σ. This is the matrix obtained from the unit matrix $\mathfrak{E}_I$ by permuting the columns according to σ : The j-th column of $\mathfrak{P}_\sigma$ is $e_{\sigma(j)}$, see Example 8.C.6. Then for $\sigma,\tau \in \mathfrak{S}(I)$:

(a) $\mathrm{Det}\,\mathfrak{P}_\sigma = \mathrm{Sign}\,\sigma$. **(b)** $\mathfrak{P}_{\sigma\tau} = \mathfrak{P}_\sigma\mathfrak{P}_\tau$. **(c)** $(\mathfrak{P}_\sigma)^{-1} = \mathfrak{P}_{\sigma^{-1}} = {}^t(\mathfrak{P}_\sigma)$.

(**Proof :** (a) Obviously, $\mathrm{Det}\,\mathfrak{P}_\sigma = (\mathrm{Sign}\sigma)\,\mathrm{Det}\,\mathfrak{E}_I = \mathrm{Sign}\sigma$ (see Rule (3) before Theorem 9.D.2).

(b) $\mathfrak{P}_\sigma = (\delta_{i,\sigma j}) = (\delta_{\sigma^{-1}i,j})$, $\mathfrak{P}_\tau = (\delta_{j,\tau k})$. The (i,k)-th entry of the matrix $\mathfrak{P}_\sigma\mathfrak{P}_\tau$ is $\sum_{j=1}^{n}\delta_{\sigma^{-1}i,j}\delta_{j,\tau k} = \delta_{\sigma^{-1}i,\tau k} = \delta_{i,\sigma\tau k}$ which is the (i,k)-th entry of the matrix $\mathfrak{P}_{\sigma\tau}$. Or : $\mathfrak{P}_\sigma$ is the matrix of the endomorphism $f_\sigma : K^I \to K^I$, $f_\sigma(e_j) = e_{\sigma(j)}$, $j\in I$, with respct to the standard basis e_i, $i\in I$, of K^I. Then $\mathfrak{P}_\sigma\mathfrak{P}_\tau$ is the matrix of the composition $f_\sigma f_\tau : e_j \mapsto e_{\tau(j)} \mapsto e_{\sigma\tau(j)}$, and hence $\mathfrak{P}_{\sigma\tau}$ is the matrix of $f_{\sigma\tau}$.

— **Remark :** *The homomorphisms* $\sigma \mapsto \mathfrak{P}_\sigma$ *and* $\sigma \mapsto f_\sigma$ *are canonical embeddings of the group* $\mathfrak{S}(I)$ *in the groups* $\mathrm{GL}_I(K)$ *and* $\mathrm{Aut}\,(K^I)$, resp.

(c) $\mathfrak{P}_\sigma\mathfrak{P}_{\sigma^{-1}} = \mathfrak{P}_{\sigma\sigma^{-1}} = \mathfrak{P}_{\mathrm{id}} = \mathfrak{E}_I$, by (b) and hence $(\mathfrak{P}_\sigma)^{-1} = \mathfrak{P}_{\sigma^{-1}}$. Moreover, (i,j)-th entry of ${}^{\rm t}\mathfrak{P}_\sigma$ is $\delta_{j,\sigma i} = \delta_{\sigma^{-1}j,i} = \delta_{i,\sigma^{-1}j}$ which is the (i,j)-th entry of $\mathfrak{P}_{\sigma^{-1}}$. •)

S10.41 Let $\mathfrak{A} = (a_{ij}) \in \mathrm{M}_I(K)$ be a *skew-symmetric* matrix (I finite indexed), i. e., ${}^{\rm t}\mathfrak{A} = -\mathfrak{A}$. If $|I|$ is odd and if $\mathrm{Char}\,K \neq 2$, i. e., $2 = 2\cdot 1_K \neq 0$ in K, then $\mathrm{Det}\,\mathfrak{A} = 0$.

(**Proof :** By Theorem 9.D.1 $\mathrm{Det}\,\mathfrak{A} = \mathrm{Det}\,{}^{\rm t}\mathfrak{A} = \mathrm{Det}\,(-\mathfrak{A}) = (-1)^{|I|}\mathrm{Det}\,\mathfrak{A} = -\mathrm{Det}\,\mathfrak{A}$, since $|I|$ is odd. It follows that $2\cdot\mathrm{Det}\,\mathfrak{A} = 0$, and hence $\mathrm{Det}\,\mathfrak{A} = 0$ because $2 \neq 0$ in K. •)

S10.42 Let $\mathfrak{A} := (a_{ij}) \in \mathrm{M}_n(\mathbb{Z})$ be the $n \times n$-matrix defined by $a_{ij} := \binom{i}{j-1}$. Compute the determinant $\operatorname{Det} \mathfrak{A}$. (**Hint :** What is $a_{ij} - a_{i-1,j}$?)

S10.43 **(a)** For two matrices $\mathfrak{A} \in \mathrm{M}_{m,n}(K)$ and $\mathfrak{B} \in \mathrm{M}_{n,m}(K)$ with $m > n$, show that $\operatorname{Det}(\mathfrak{A}\mathfrak{B}) = 0$. (**Hint :** Consider $\mathfrak{A}$ and $\mathfrak{B}$ in $\mathrm{M}_{m,m}(K)$ by filling the extra entries 0.)
(b) Let $\mathfrak{A} = (a_{ij}) \in \mathrm{M}_n(K)$ and $\mathfrak{B} := (b_{ij}) \in \mathrm{M}_n(K)$ with $b_{ij} := (-1)^{i+j} a_{ij}$, $1 \le i, j \le n$. Show that $\operatorname{Det} \mathfrak{A} = \operatorname{Det} \mathfrak{B}$.

S10.44 Let K be a field and $\mathfrak{A} \in \mathrm{M}_r(K)$, $\mathfrak{B} \in \mathrm{M}_s(K)$, $\mathfrak{C} \in \mathrm{M}_{r,s}(K)$ and $0_{sr} = 0\mathrm{M}_{s,,r}(K)$. Then

$$\operatorname{Det} \begin{pmatrix} \mathfrak{C} & \mathfrak{A} \\ \mathfrak{B} & 0_{sr} \end{pmatrix} = (-1)^{rs} \operatorname{Det} \mathfrak{A} \cdot \operatorname{Det} \mathfrak{B}\,.$$

(**Hint :** Each of the last r columns of the matrix have interchanged with the first s columns and hence altogether there are rs interchanges of columns and then apply the Block Matrix Theorem 9.D.4 :

$$\operatorname{Det} \begin{pmatrix} \mathfrak{C} & \mathfrak{A} \\ \mathfrak{B} & 0_{sr} \end{pmatrix} = (-1)^{rs} \operatorname{Det} \begin{pmatrix} \mathfrak{A} & \mathfrak{C} \\ 0_{sr} & \mathfrak{B} \end{pmatrix} = (-1)^{rs} \operatorname{Det} \mathfrak{A} \cdot \operatorname{Det} \mathfrak{B}\,.)$$

S10.45 Prove the Product Formula 9.D.5 for determinants as follows :
Let $\mathfrak{A}, \mathfrak{B} \in \mathrm{M}_n(K)$. By adding suitable multiples of the first n columns of the block-matrix

$$\begin{pmatrix} \mathfrak{A} & 0 \\ -\mathfrak{E} & \mathfrak{B} \end{pmatrix}$$

to the last n columns transform this matrix to the block matrix

$$\begin{pmatrix} \mathfrak{A} & \mathfrak{A}\mathfrak{B} \\ -\mathfrak{E} & 0 \end{pmatrix}$$

and then use Supplement S10.44.

S10.46 Let $n \in \mathbb{N}$ be an odd natural number and $\mathfrak{A} \in \mathrm{M}_n(\mathbb{R})$. Then there exists a real number $t \in \mathbb{R}$ such that $\operatorname{Det}(\mathfrak{A} + t\mathfrak{E}_n) = 0$. (**Hint :** The determinant is a polynomial function of odd degree n in t and hence by the Intermediate Value Theorem (see Footnote 4 in Exercise Set 10) has a zero in $\mathbb{R}$. — **Remark :** Note that $\operatorname{Det}(\mathfrak{A} + t\mathfrak{E}_n) = \chi_{-\mathfrak{A}}(t)$ is the characteristic polynomial $\chi_{-\mathfrak{A}}$ of $-\mathfrak{A}$, see Subsection 11.A)

S10.47 Let $f_1, \ldots, f_n$ functions on the set D with values in the field K. Then show that $f_1, \ldots, f_n$ are linearly independent in K^D if and only if the function

$$(t_1, \ldots, t_n) \longmapsto \begin{vmatrix} f_1(t_1) & \cdots & f_1(t_n) \\ \vdots & \ddots & \vdots \\ f_n(t_1) & \cdots & f_n(t_n) \end{vmatrix}$$

on D^n is not the zero-function. (**Remark :** See Theorem 5.G.17 — Determinants of this form are called a l t e r n a n t or (particularly in Physics) S l a t e r ' s D eterminant. For example the Vandermonde's determinant corresponding to $f_i := t^{i-1}$, $i = 1, \ldots, n$, $D := K$, see the Exercise 10.6 (a) and the Cauchy's double-alternants, see the Exercise 9.5-(b).

S10.48 Let $f_1, \ldots, f_n$ be polynomial functions over K of deg $< n-1$, $n \in \mathbb{N}^*$. For all $t_1, \ldots, t_n \in K$, prove that :

$$\begin{vmatrix} f_1(t_1) & \cdots & f_1(t_n) \\ \vdots & \ddots & \vdots \\ f_n(t_1) & \cdots & f_n(t_n) \end{vmatrix} = 0\,.$$

S10.49 (C a u c h y ' s D o u b l e - a l t e r n a n t) Let $a_1, \ldots, a_n, b_1, \ldots, b_n \in K$ with $a_i + b_j \neq 0$ for all $i, j = 1, \ldots, n$. Show that

$$\operatorname{Det}\left(\left(\frac{1}{a_i + b_j} \right)_{1 \le i,j \le n} \right) = \frac{\prod_{1 \le i < j \le n} (a_j - a_i) \prod_{1 \le i < j \le n} (b_j - b_i)}{\prod_{i,j=1}^{n} (a_i + b_j)}\,.$$

(**Hint :** Induction on n. — See also Supplement S9.22.)

S10.50 For $t_1,\ldots,t_n, u_1,\ldots,u_n \in \mathbb{C}$, compute

$$\begin{vmatrix} \sin(t_1+u_1) & \sin(t_1+u_2) & \cdots & \sin(t_1+u_n) \\ \sin(t_2+u_1) & \sin(t_2+u_2) & \cdots & \sin(t_2+u_n) \\ \vdots & \vdots & \ddots & \vdots \\ \sin(t_n+u_1) & \sin(t_n+u_2) & \cdots & \sin(t_n+u_n) \end{vmatrix}.$$

(**Hint:** The two cases $n \leq 2$ and $n > 2$ separately. For $n \geq 3$, we apply the addition theorem for the sin function and the Determinant product formula to note that

$$D_n = \begin{vmatrix} \sin t_1 & \cos t_1 & 0 & \cdots & 0 \\ \sin t_2 & \cos t_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sin t_n & \cos t_n & 0 & \cdots & 0 \end{vmatrix} \begin{vmatrix} \cos u_1 & \cos u_2 & \cdots & \cos u_n \\ \sin u_1 & \sin u_2 & \cdots & \sin u_n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{vmatrix} = 0 \cdot 0 = 0.$$

See also Supplement S10.17.)

S10.51 For elements $a_1,\ldots,a_n,b_1,\ldots b_n$, $n \in \mathbb{N}^*$, of a field K, show that:

$$D_n := \begin{vmatrix} 1+a_1b_1 & 1+a_1b_2 & \cdots & 1+a_1b_n \\ 1+a_2b_1 & 1+a_2b_2 & \cdots & 1+a_2b_n \\ \vdots & \vdots & \ddots & \vdots \\ 1+a_nb_1 & 1+a_nb_2 & \cdots & 1+a_nb_n \end{vmatrix} = 0,$$

if $n \geq 3$, and $D_1 = 1+a_1b_1$, $D_2 = (a_2-a_1)(b_2-b_1)$.

S10.52 Let D be a set, $t_1,\ldots,t_n \in D$ and $f_0,\ldots,f_n$ be linearly independent K-valued functions on D such that the $(n+1) \times n$-matrix

$$\begin{pmatrix} f_0(t_1) & \cdots & f_0(t_n) \\ \vdots & \ddots & \vdots \\ f_n(t_1) & \cdots & f_n(t_n) \end{pmatrix}$$

has the maximal rank n. (because of the linear independence of $f_0,\ldots,f_n$, this is the case in general, see Supplement S10.47. In this case we say that the points $t_1,\ldots,t_n$ are in *general position* with respect to the $f_0,\ldots,f_n$.) Then show that the function

$$t \longmapsto \begin{vmatrix} f_0(t) & f_0(t_1) & \cdots & f_0(t_n) \\ f_1(t) & f_1(t_1) & \cdots & f_1(t_n) \\ \vdots & \vdots & \ddots & \vdots \\ f_n(t) & f_n(t_1) & \cdots & f_n(t_n) \end{vmatrix}$$

is a non-trivial linear combination of the functions $f_0,\ldots,f_n$, which vanish on the points $t_1,\ldots,t_n$ and is uniquely determined up to a constant factor $\lambda \neq 0$,

S10.53 Let D be a set, $E := \{t_1,\ldots,t_n\}$ be a subset of D with n elements and $f_1,\ldots,f_n$ K-valued functions on D with

$$\begin{vmatrix} f_1(t_1) & \cdots & f_1(t_n) \\ \vdots & \ddots & \vdots \\ f_n(t_1) & \cdots & f_n(t_n) \end{vmatrix} \neq 0.$$

Show that the functions $f_1|_E,\ldots,f_n|_E$ form a basis of K^E. For arbitrary elements $b_1,\ldots,b_n \in K$, there exists a unique linear combination f of $f_1,\ldots,f_n$ with $f(t_i) = b_i$, $i = 1,\ldots,n$. This follows from the equation

$$\begin{vmatrix} f(t) & b_1 & \ldots & b_n \\ f_1(t) & f_1(t_1) & \cdots & f_1(t_n) \\ \vdots & \vdots & \ddots & \vdots \\ f_n(t) & f_n(t_1) & \cdots & f_n(t_n) \end{vmatrix} = 0$$

by expanding in terms of the first column. (**Remark:** The uniquely determined function f is called the solution of the *interpolation problem* $f(t_i) = b_i$, $i = 1,\ldots,n$, *with the functions* $f_1,\ldots,f_n$.)

S10.54 Let $f \in \mathbb{Z}^{\mathbb{N}^*}$ be number-theoretic function and let $F \in \mathbb{Z}^{\mathbb{N}^*}$ be its *summator function of f*, i.e., $F(n) := \sum_{d|n} f(d)$, $n \in \mathbb{N}^*$. For $n \in \mathbb{N}^*$, show that the determinant of the matrix $\mathfrak{F} := \big(F(\gcd(i,j))\big)_{1\leq i,j\leq n} \in \mathrm{M}_n(\mathbb{Z})$ is equal the product $\prod_{m=1}^{n} f(m)$. In particular, (*Formula of Henry J. S. Smith*[7]) $\mathrm{Det}\,\mathfrak{F} = \varphi(1)\cdots\varphi(n) = n!\prod_{p\in\mathbb{P}}\left(1-\frac{1}{p}\right)^{[n/p]}$, where φ is the Euler's totient function, see Supplement S1.3. (**Hint:** For the computation of $\mathrm{Det}\,\mathfrak{F}$, we consider the matrix $\mathfrak{M} := (\mu_{ij})_{1\leq i,j\leq n}$, where $\mu_{ij} := \begin{cases}\mu(i/j) & \text{if } j \text{ divides } i\,,\\ 0, & \text{otherwise}.\end{cases}$, and $\mu : \mathbb{N}^* \to \mathbb{Z}$ is the *Möbius function*[8] defined by $\mu(n) := (-1)^r$, if $n = p_1\cdots p_r$ is the product of r distinct prime numbers, otherwise $\mu(n) = 0$. Note that $\mathfrak{M}$ is a lower triangular matrix and $\mathfrak{M}\mathfrak{F}$ is an upper triangular matrix with diagonal entries $f(1),\ldots,f(n)$. This follows immediately from the so-called *Möbius inversion formula*: (a relation between a number theoretic function and its summator function) $f(m) = \sum_{d|m} \mu(m/d)\cdot F(d)$, $m \in \mathbb{N}^*$. The last formula of Smith follows from the fact that the summator function of the Euler's totient function φ is the function $\psi : \mathbb{N}^* \to \mathbb{Z}$, $n \mapsto n$, since $n = \sum_{d|n}\varphi(d)$.

— **Remarks:** It is interesting to note that number-theoretic functions and their properties can be studied lucidly by using the ring structure on $\mathbb{Z}^{\mathbb{N}^*}$, where addition is defined point-wise and the multiplication is defined using so-called *Dirichlet's convolution*: For $f, g \in \mathbb{Z}^{\mathbb{N}*}$, define $(f*g)(n) := \sum_{d|n} f(d)g(n/d)$. With these addition and multiplication $\mathbb{Z}^{\mathbb{N}*}$ is a commutative ring — called the *ring of number-theoretic functions* denoted by $\mathrm{ZF}(\mathbb{Z})$ and its elements are called *number-theoretic functions*. The multiplicative indenty in this ring is the function $\varepsilon : \mathbb{N}^* \to \mathbb{Z}$, defined by $\varepsilon(1) = 1$ and $\varepsilon(n) = 0$ for $n \geq 2$. An element $e \in \mathrm{ZF}(\mathbb{Z})$ is a unit if and only if $e(1) \in \mathbb{Z}^\times = \{\pm 1\}$. Euler's totient function φ, the functions $\mathrm{T}, \mathrm{S} : \mathbb{N}^* \to \mathbb{Z}$ with $\mathrm{T}(n)$ (resp. $\mathrm{S}(n)$) the number of positive divisors (resp. the sum of positive divisors) of n, the function $\zeta : \mathbb{N}^* \to \mathbb{Z}$, $\zeta(n) := 1$ for all $n \in \mathbb{N}^*$ are all number-theoretic functions studied in elementary number theory. It is easy to check that $\zeta * f$ is the summator function of every $f \in \mathrm{ZF}(\mathbb{Z})$; $\zeta*\zeta = T$, $\zeta*\psi = S$. Further, $\zeta \in \mathrm{ZF}(\mathbb{Z})^\times$ and $\zeta^{-1} = \mu$ is the Möbius function defined above and hence $f = \mu*(\zeta*f)$ for every $f \in \mathrm{ZF}(\mathbb{Z})$.)

S10.55 **(a)** Let $P_i = (a_{1i},\ldots,a_{ni})$, $i = 0,\ldots,n$ be points in the affine space $\mathbb{A}^n(K) = K^n$. Then the P_i are affinely dependent if and only if

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_{10} & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n0} & a_{n1} & \cdots & a_{nn} \end{vmatrix}.$$

(b) Let $P_i = (a_{1i},\ldots,a_{ni})$, $i = 1,\ldots,n$ be affinely independent points in $\mathbb{A}^n(K) = K^n$. The equation of the affine hyperplane H in $\mathbb{A}^n(K)$ generated by the points $P_1,\ldots,P_n$ is

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n & a_{n1} & \cdots & a_{nn} \end{vmatrix} = 0\,,$$

i. e., the point $P = (x_1,\ldots,x_n) \in K^n$ belong to H if and only if its component satisfy the above (affine) equation. (See Supplement S9.36.)

[7]Henry John Stephen *Smith* (1826 – 1883) was an Irish mathematician remembered for his work in elementary divisors, quadratic forms, and Smith-Minkowski-Siegel mass formula in number theory. In matrix theory the *Smith Normal Form* a normal form that can be defined for any matrix (not necessarily square) with entries in a principal ideal domain (PID), e. g. $\mathbb{Z}$, it is a diagonal matrix, and can be obtained from the original matrix by multiplying on the left and right by invertible square matrices. In particular, since $\mathbb{Z}$ is a PID, so one can always calculate the Smith normal form of an integer matrix. The Smith normal form is very useful for working with finitely generated modules over a PID, and in particular for deducing the structure of a quotient of a free module.

[8]In 1832 *A. F. Möbius* (1790 – 1868) defined Möbius function which is important in number theory and combinatorics where it is used and generalized extensively.

S10.56 Let $P_1=(a_{11},a_{21})$, $P_2=(a_{12},a_{22})$, $P_3=(a_{13},a_{23})$ be three points in $\mathbb{R}^2$ which do not lie on a line. Then show that :

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & a_{11} & a_{12} & a_{13} \\ x_2 & a_{21} & a_{22} & a_{23} \\ x_1^2+x_2^2 & a_{11}^2+a_{21}^2 & a_{12}^2+a_{22}^2 & a_{13}^2+a_{23}^2 \end{vmatrix} = 0$$

is the equation of the circle passing through P_1, P_2, P_3.

S10.57 Let (a_{ij}) and (b_{ij}) be two $n\times n$-matrices over the field K. Then show that :

$$\sum_{i=1}^{n} \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ b_{i1} & \cdots & b_{in} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn}[\end{vmatrix} = \sum_{j=1}^{n} \begin{vmatrix} a_{11} & \cdots & b_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & b_{nj} & \cdots & a_{nn} \end{vmatrix}.$$

(**Hint :** If $(-1)^{i+j}A_{ij}$ are the cofactors of (a_{ij}), then by expanding the determinants by using the i-th row respectively the j-th column we have the equality :

$$\sum_{i=1}^{n} \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ b_{i1} & \cdots & b_{in} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = \sum_{i=1}^{n}\sum_{j=1}^{n}(-1)^{i+j}b_{ij}A_{ij} = \sum_{j=1}^{n}\sum_{i=1}^{n}(-1)^{i+j}b_{ij}A_{ij} = \sum_{j=1}^{n} \begin{vmatrix} a_{11} & \cdots & b_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & b_{nj} & \cdots & a_{nn} \end{vmatrix}.)$$

S10.58 Compute the following $n\times n$-determinants over $\mathbb{Q}$:

(**a**) $$\begin{vmatrix} 1 & n & n & \cdots & n \\ n & 2 & n & \cdots & n \\ n & n & 3 & \cdots & n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n & n & \cdots & n \end{vmatrix}.$$

Ans $:= (-1)^{n-1}n!$

(**b**) $$\begin{vmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 2 & 1 & 2 & 3 & \cdots & n-1 \\ 3 & 2 & 1 & 2 & \cdots & n-2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n-1 & n-2 & n-3 & \cdots & 1 \end{vmatrix}.$$

Ans $:= (-1)^{n-1}(n+1)2^{n-1}$

(**c**) $$\begin{vmatrix} 1 & 2 & 2 & \cdots & 2 \\ 2 & 2 & 2 & \cdots & 2 \\ 2 & 2 & 3 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & 2 & \cdots & n \end{vmatrix}.$$

Ans $:= (-2)(n-2)!$

(**d**) $$\begin{vmatrix} 1 & 2 & 3 & \cdots & n-2 & n-1 & n \\ 2 & 3 & 4 & \cdots & n-1 & n & 1 \\ 3 & 4 & 5 & \cdots & n & 1 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ n & 1 & 2 & \cdots & n-3 & n-2 & n-1 \end{vmatrix}.$$

Ans $:= -1)^{\binom{n}{2}}(n+1)n^{n-1}/2$

(**Hints** (a) Subtract the blast column from all other columns to get the upper triangular matrix with diagonal entries $1-n, 2-n, 3-n, \ldots, 1, n$.)

S10.59 Let $n\in\mathbb{N}$, $n\geq 2$. Compute the determinant of the following matrices from $\mathrm{M}_n(\mathbb{Z})$:

(**a**) $$\begin{vmatrix} 1 & 2 & \cdots & n \\ n+1 & n+2 & \cdots & 2n \\ 2n+1 & 2n+2 & \cdots & 3n \\ \vdots & \vdots & \vdots & \vdots \\ (n-1)n+1 & (n-1)n+2 & \cdots & n^2 \end{vmatrix}.$$

(**b**) $$\begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 1 & 1 & 1 & \cdots & 1 & 1-n \\ 1 & 1 & 1 & \cdots & 1-n & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1-n & 1 & \cdots & 1 & 1 \end{vmatrix}.$$

(**Hint :** For the matrix (b) add all other columns to the first column and then successively interchange 1-st column with n-th, 2-nd with $(n-1)$-th etc. and apply Supplement S10.60 (a). — **Ans :** $(-1)^{\binom{n}{2}}\,\frac{n+1}{2}\,n^{n-1}$.)

(c) $$\begin{vmatrix} 1 & n & n & \cdots & n \\ n & 2 & n & \cdots & n \\ n & n & 3 & \cdots & n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n & n & \cdots & n \end{vmatrix}.$$

S10.60 Verify the following determinant formulas for $(n+1)\times(n+1)$-matrices with coefficients in a field K. (At the places marked by $*$ one may take arbitrary elements of K.)

(a) $$\begin{vmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & a \end{vmatrix} = (a+nb)(a-b)^n .$$

(b) $$\begin{vmatrix} 1 & a_1 & a_2 & \cdots & a_n \\ 1 & a_1+b_1 & * & \cdots & * \\ 1 & a_1 & a_2+b_2 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_1 & a_2 & \cdots & a_n+b_n \end{vmatrix} = b_1\cdots b_n .$$

(c) $$\begin{vmatrix} a_1 & * & * & * & \cdots & * & 1 \\ b_1 & a_2 & * & * & \cdots & * & 1 \\ b_1 & b_2 & a_3 & * & \cdots & * & 1 \\ b_1 & b_2 & b_3 & a_4 & \cdots & * & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_1 & b_2 & b_3 & b_4 & \cdots & a_n & * \\ b_1 & b_2 & b_3 & b_4 & \cdots & b_n & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ b_1 & a_1 & a_1 & a_1 & \cdots & a_1 & a_1 \\ * & b_2 & a_2 & a_2 & \cdots & a_2 & a_2 \\ * & * & b_3 & a_3 & \cdots & a_3 & a_3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & * & \cdots & a_{n-1} & a_{n-1} \\ * & * & * & * & \cdots & b_n & a_n \end{vmatrix} = (a_1-b_1)\cdots(a_n-b_n) .$$

(d) $$\begin{vmatrix} -a_1 & a_1 & 0 & \cdots & 0 & 0 \\ 0 & -a_2 & a_2 & \cdots & 0 & 0 \\ 0 & 0 & -a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -a_n & a_n \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{vmatrix} = (-1)^n(n+1)\,a_1\cdots a_n .$$

S10.61 Prove the following determinant formulas for the $n\times n$-matrices over a field K: Let $a_1,\ldots,a_n$, $b_1,\ldots,b_n$, $c_1,\ldots,c_{n-1}$ be elements of K and let

$$D_n := \begin{vmatrix} a_1 & b_1 & 0 & \cdots & 0 & 0 \\ c_1 & a_2 & b_2 & \cdots & 0 & 0 \\ 0 & c_2 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & \cdots & c_{n-1} & a_n \end{vmatrix}$$

(a) (Recursion formula) : $D_k = a_kD_{k-1} - b_{k-1}c_{k-1}D_{k-2}$, for all $k = 2,\ldots,n$.

(b) In part (a) put $b_1 = \cdots = b_{n-1} = c_1 = \cdots c_{n-1} =: b$ and $D_n := D(b\,; a_1,\ldots,a_n)$. Then
$$D(b\,; a_1,\ldots,a_n) = a_n\,D(b\,; a_1,\ldots,a_{n-1}) - b^2\,D(b\,; a_1,\ldots,a_{n-2}) \text{ for all } n\geq 2.$$

(c) Compute the determinant $D(b\,; a_1,\ldots,a_n)$ in the following cases :

(1) $b = a_1 = \cdots = a_n = 1$.
(2) $a_1 = \cdots = a_n = 0$.

(3) $K = \mathbb{K}$ and $b = 1$, $a_1 = \cos\varphi$, $a_2 = \cdots = a_n = 2\cos\varphi$.

$$\begin{vmatrix} \cos\varphi & 1 & 0 & \cdots & 0 \\ 1 & 2\cos\varphi & 1 & \cdots & 0 \\ 0 & 1 & 2\cos\varphi & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2\cos\varphi \end{vmatrix} = \cos n\varphi\,, \quad \varphi \in \mathbb{C}.$$

(**Remark:** For the modified Tchebychev Polynomial $\widetilde{T}_n$ see the recursion-formula in (3)-(iii) below. — Recall the definition and some properties of **Tchebychev Polynomials:**

For $n \in \mathbb{N}$ the polynomials

$$T_n(X) := \sum_{k=0}^{[n/2]} \left(-\frac{1}{4}\right)^k \frac{n}{n-k}\binom{n-k}{k} X^{n-2k} \text{ and } U_n(X) := \sum_{k=0}^{[n/2]} \left(-\frac{1}{4}\right)^k \binom{n-k}{k} X^{n-2k}$$

are called Tchebychev polynomials of first and second kind respectively.

Properties of Tchebychev polynomials.

(1) $T_0 = 2, T_1 = X$ and $T_{n+2} = XT_{n+1} - \frac{1}{4}T_n$ for every $n \in \mathbb{N}$.

(2) $2^{n-1}T_n(\cos(\varphi)) = \cos(n\varphi)$ for every $n \in \mathbb{N}$ and $\varphi \in \mathbb{R}$.

(3) For $n \in \mathbb{N}$, put $\widetilde{T}_n(X) := 2^{n-1}T_n(X)$. Then:

(i) $\widetilde{T}_0 = 1, \widetilde{T}_1 = X$ and $\widetilde{T}_{n+2} = 2X\widetilde{T}_{n+1} - \widetilde{T}_n$ for every $n \in \mathbb{N}$.

(ii) Let $n \in \mathbb{N}$. Then $\widetilde{T}_n(1) = 1, \widetilde{T}_n(-1) = (-1)^n$ and $\widetilde{T}_n(0) = \begin{cases} (-1)^{n/2} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$

(iii) $\widetilde{T}_n(\cos(\varphi)) = \cos(n\varphi)$ for every $n \in \mathbb{N}$ and $\varphi \in \mathbb{R}$.

(4) T_n and $\widetilde{T}_n$ have n-distinct real zeros in the open interval $(-1,1)$, namely : $\cos((2k+1)\pi/2n)$ for $k = 0, \dots, n-1$ and therefore $T_n(X) = \prod_{k=0}^{n-1}\Big(X - \cos((2k+1)\pi/2n)\Big)$ for every $n \geq 1$.)

(4) $a_1 = \cdots = a_n =: a$.

$$\begin{vmatrix} a & b & 0 & \cdots & 0 & 0 \\ b & a & b & \cdots & 0 & 0 \\ 0 & b & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & \cdots & b & a \end{vmatrix} = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} a^{n-2k} b^{2k}.$$

(d) In part (a) put $b_1 = \cdots = b_{n-1} = -c_1 = \cdots - c_{n-1} =: b$ and $D_n := \Delta(b; a_1, \dots, a_n)$. Then

$$\Delta(b; a_1, \dots, a_n) = a_n\Delta(b; a_1, \dots, a_{n-1}) - b^2\Delta(b; a_1, \dots, a_{n-2}) \text{ for all } n \geq 2.$$

Further, for $a_1 = \cdots = a_n =: a$,

$$\Delta(b; a, \dots, a) = \begin{vmatrix} a & b & 0 & \cdots & 0 & 0 \\ -b & a & b & \cdots & 0 & 0 \\ 0 & -b & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & \cdots & -b & a \end{vmatrix} = \sum_{k=0}^{[n/2]} \binom{n-k}{k} a^{n-2k} b^{2k}.$$

(**Remark:** For $a = b = 1$, the determinant $\Delta(1; 1, \dots, 1)$ is the Fibonacci-number f_{n+1} (the $n+1$-term in the Fibonacci sequence $f_0 := 0, f_1 : +1, f_n := f_{n-1} + f_{n-2}$ for $n \geq 2$), which is equal to (Binet's formula): $f_{n+1} := \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)$. See also Supplement S1.8.)

S10.62 Compute the determinants of the following matrices in $\mathrm{M}_n(\mathbb{Z})$:

(**a**) $$\begin{vmatrix} 2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 2 \end{vmatrix}.$$

(**Hint :** Use induction on n. See also Supplement S10.61 (c) (4) ($a=2$ $b=1$).)

(**b**) $$\begin{vmatrix} 1 & 1^2 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & 1 & 2^2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 3^2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 & (n-2)^2 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 1 & (n-1)^2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 1 \end{vmatrix}.$$

(**Hint :** Use induction on n and recursion formula in Supplement S10.61 (c) (4) ($a=1$, $b_i=i^2$, $i=1,\ldots,n-1$, and $c_1=c_2=\cdots=c_{n-1}=1$).)

(**c**) $$\begin{vmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{vmatrix}$$

(**Hint :** Expand using the first two columns and use Supplement S10.61 (d) ($a=2$ and $b=1$).)

S10.63 Let $a_1,\ldots,a_n,b$ and a_{ij}, $1\leq i,j\leq n$ be elements of a field K. Then show that :

(**a**) $$\begin{vmatrix} a_0+a_1 & a_1 & 0 & \cdots & 0 \\ a_1 & a_1+a_2 & a_2 & \cdots & 0 \\ 0 & a_2 & a_2+a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1}+a_n \end{vmatrix} = \sum_{k=0}^{n}\Big(\prod_{i\neq k} a_i\Big).$$

(**b**) $$\begin{vmatrix} a_{11}+b & a_{12}+b & \cdots & a_{1n}+b \\ a_{21}+b & a_{22}+b & \cdots & a_{2n}+b \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}+b & a_{n2}+b & \cdots & a_{nn}+b \end{vmatrix} = a + b\left(\sum_{i,j=1}^{n} a'_{ij}\right),$$

where $a:=\operatorname{Det}\big(a_{ij}\big)$ and a'_{ij} is the (i,j)-th cofactor of $\big(a_{ij}\big)$, $1\leq i,j\leq n$.

S10.64 Prove the following determinant formulas by induction :

(**a**) $$\begin{vmatrix} a_1+b_1 & b_1 & b_1 & \cdots & b_1 \\ b_2 & a_2+b_2 & b_2 & \cdots & b_2 \\ b_3 & b_3 & a_3+b_3 & \cdots & b_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_n & b_n & b_n & \cdots & a_n+b_n \end{vmatrix} = a_1\cdots a_n + \sum_{k=1}^{n}\Big(\prod_{i\neq k} a_i\Big) b_k,$$

(b) $$\begin{vmatrix} x+a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\ -1 & x & 0 & \cdots & 0 & 0 \\ 0 & -1 & x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x & 0 \\ 0 & 0 & 0 & \cdots & -1 & x \end{vmatrix} = x^n + a_1 x^{n-1} + \cdots + a_n \,,.$$

(c) $$\begin{vmatrix} a_1 & \cdots & 0 & 0 & \cdots & b_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & a_n & b_n & \cdots & 0 \\ 0 & \cdots & b_n & a_n & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ b_1 & \cdots & 0 & 0 & \cdots & a_1 \end{vmatrix} = \prod_{k=1}^{n} (a_k^2 - b_k^2)\,.$$

S10.65 Compute the determinant of the $n \times n$ matrix over a field K:

(a) $$\begin{vmatrix} 1+a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ a_2b_1 & 1+a_2b_2 & \cdots & a_2b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_nb_1 & a_nb_2 & \cdots & 1+a_nb_n \end{vmatrix}\,.$$

(**Hint:** If all $a_i = 0$, then it is the identity matrix and hence its determinant is 1. Otherwise, we may assume that $a_n \neq 0$. For $i = 1, \ldots, n-1$, replace i-th row by adding $-a_i a_n^{-1}$-times the n-th row to it and then replace the last row by by adding the $-a_n b_i$-times the i-th row, we get an upper triangular matrix :

$$\begin{vmatrix} 1+a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ a_2b_1 & 1+a_2b_2 & \cdots & a_2b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_nb_1 & a_nb_2 & \cdots & 1+a_nb_n \end{vmatrix} = \begin{vmatrix} 1 & 0 & \cdots & -a_1a_n^{-1} \\ 0 & 1 & \cdots & -a_2a_n^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_nb_1 & a_nb_2 & \cdots & 1+a_nb_n \end{vmatrix} = \begin{vmatrix} 1 & 0 & \cdots & -a_1a_n^{-1} \\ 0 & 1 & \cdots & -a_2a_n^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1+\sum_{i=1}^{n} a_ib_i \end{vmatrix} = 1 + \sum_{i=1}^{n} a_ib_i\,.)$$

(b) Solve the following system of linear equations by using Cramer's rule :

$$\begin{aligned} x_2 + x_3 + \cdots + x_{n-1} + x_n &= 1 \\ x_1 \quad\;\; + x_3 + \cdots + x_{n-1} + x_n &= 1 \\ x_1 + x_2 \quad\;\; + \cdots + x_{n-1} + x_n &= 1 \\ \cdots\cdots \quad \cdots \quad \cdots\cdots \quad\quad & \cdots \\ x_1 + x_2 + x_3 + \cdots + x_{n-1} + x_n &= 1 \end{aligned}$$

(**Hint:** Clearly, one sees immediately that $x_k + 1/(n-1)$, $k = 1, \ldots, n$, is a solution. The Cramer's Rule 9.D.14 shows that $x_k = D_k/D$ if the the denominator determinant $D = \mathrm{Det}\,(\mathfrak{J}_n - \mathfrak{E}_n) \neq 0$, where $\mathfrak{J}_n$ is the matrix in the Supplement S10.35 (b). For its computation, we use Supplement S10.60 (a) with n instead of $n+1$, $a = 0$ and $b = 1$ and note that $D = (-1)^{n-1}(n-1) \neq 0$. For the computation of the numerator determinants $D_k = \mathrm{Det}\,(\mathfrak{J} - \mathfrak{E}_n - \mathfrak{E}_{kk})$, first subtract k-th column from all other columns and then all other columns to the k-th column to get the diagonal matrix $-\mathfrak{E}_n + 2\cdot\mathfrak{E}_{kk}$ and hence $D_k = \mathrm{Det}\,(-\mathfrak{E}_n + 2\mathfrak{E}_{kk}) = (-1)^{n-1}$. Therefore, we have again proved that $x_k = D_k/D = 1/(n-1)$, $k = 1, \ldots, n$. — One can also compute the values of $D, D_1, \ldots, D_n$ by directly using the Remark in Supplement S10.35.)

S10.66 Suppose that the matrix $\mathfrak{A} = (a_{ij}) \in \mathrm{GL}_n(K)$ satisfy the hypothesis of Supplement S9.41 and suppose that $\mathfrak{A} = \mathfrak{L}\mathfrak{D}\mathfrak{R}'$ with a diagonal matrix $\mathfrak{D} = \mathrm{Diag}\,(a_1, \ldots, a_n)$ and a normalised lower respectively upper triangular matrix $\mathfrak{L}$ respectively $\mathfrak{R}'$. Then $a_k = D_k/D_{k-1}$, $k = 1, \ldots, n$, where $D_k = \mathrm{Det}\,(a_{ij})_{1 \le i,j \le k}$ is the k-th principal minor of $\mathfrak{A}$, $k = 0, \ldots, n$. (Put $D_0 = 1$.)

S10.67 Let $n \in \mathbb{N}^*$ and let K be a field. The canonical exact sequence

$$1 \longrightarrow \mathrm{SL}_n(K) \longrightarrow \mathrm{GL}_n(K) \xrightarrow{\mathrm{Det}} K^\times \longrightarrow 1$$

is a weak-split. Further, it is strong-split if and only if the power-map $x \mapsto x^n$ is an automorphism of $K^\times$. (**Remarks :** An *exact sequence* (i. e., (i) φ is injective, (ii) ψ is surjective and (iii) $\mathrm{Im}\,\varphi = \mathrm{Ker}\,\psi$.)

(*) $$1 \to N \xrightarrow{\varphi} G \xrightarrow{\psi} H \to 1$$

of groups (not necessary abelian) is called a w e a k s p l i t s e q u e n c e if ψ has a *section* σ, i. e. there exists a homomorphism $\sigma : H \to G$ such that $\psi\sigma = \mathrm{id}_H$ (this means G is the *semi-direct product* of $\mathrm{Im}\,\varphi \cong N$ and $\mathrm{Im}\,\sigma \cong H$) and $\mathrm{Im}\,\sigma$ is called a w e a k c o m p l e m e n t of $\mathrm{Im}\,\varphi$ in G. — If there exists a projection $\pi : G \to N$ such that $\pi\varphi = \mathrm{id}_N$, then G is a direct product of $\mathrm{Im}\,\varphi \cong N$ and $\mathrm{Ker}\,\pi \cong H$, i. e. the map $\mathrm{Im}\,\varphi \times \mathrm{Ker}\,\pi \to G$, $(x,y) \mapsto xy$ is an isomorphism of groups. In this we say that the exact sequence $(*)$ is a s t r o n g s p l i t s e q u e n c e and $\mathrm{Ker}\,\pi$ is called a s t r o n g c o m p l e m e n t of $\mathrm{Im}\,\varphi$ in G. – Every strong split sequence is a weak split sequence. If σ is a section of ψ and if $\mathrm{Im}\,\sigma$ is a *normal* in G, then $\mathrm{Im}\,\sigma$ is a strong complement if $\mathrm{Im}\,\varphi$ in G and the exact sequence $(*)$ is a strong split. —- If G (and hence H and N are abelian) then an exact sequence $(*)$ is weak split if and only if its strong split.)

S10.68 Let $f : V \to V$ be a nilpotent endomorphism of the n-dimensional K-vector space V. Then show that $\mathrm{Det}\,(a \cdot \mathrm{id}_V + f) = a^n$ for all $a \in K$. More generally, show that $\mathrm{Det}\,(g + f) = \mathrm{Det}\,g$ for every operator g on V which commute with f, i. e., $gf = fg$.

S10.69 Let $V := K[t]$ be the vector space of all polynomial functions over the infinite field K and let $V_n := K[t]_n$ be the subspace of all polynomial functions of degree $< n$, $n \in \mathbb{N}^*$.

(a) For $a, b \in K$, let $\varepsilon : V \to V$ be defined by $f(t) \mapsto f(at+b)$. Show that ε linear and $\varepsilon(V_n) \subseteq V_n$ for all n. Further, compute the determinant $\mathrm{Det}\,(\varepsilon\,|_{V_n})$.

(b) Let $K = \mathbb{K}$. For $c_0, \ldots, c_r \in \mathbb{K}$, let $\delta : V \to V$ be the differential operator

$$f(t) \mapsto \sum_{k=0}^{r} c_k f^{(k)}(t)\,.$$

Show that δ linear and for every $n \in \mathbb{N}^*$, $\delta(V_n) \subseteq V_n$. Further, compute the determinant $\mathrm{Det}\,(\delta\,|_{V_n})$.

S10.70 Let $m, n \in \mathbb{N}$ with $m \le n$. For arbitrary matrices $\mathfrak{A} = (a_{ij}) \in \mathrm{M}_{m,n}(K)$ and $\mathfrak{B} = (b_{ji}) \in \mathrm{M}_{n,m}(K)$ over a field K, show that

$$\mathrm{Det}\,(\mathfrak{A}\mathfrak{B}) = \sum_{1 \le j_1 < \cdots < j_m \le n} \begin{vmatrix} a_{1,j_1} & \cdots & a_{1,j_m} \\ \vdots & \ddots & \vdots \\ a_{m,j_1} & \cdots & a_{m,j_m} \end{vmatrix} \cdot \begin{vmatrix} b_{j_1,1} & \cdots & b_{j_1,m} \\ \vdots & \ddots & \vdots \\ b_{j_m,1} & \cdots & b_{j_m,m} \end{vmatrix}$$

(**Hint :** Let $f : K^n \to K^m$ and $g : K^m \to K^n$ be the linear maps defined by the matrices $\mathfrak{A}$ and $\mathfrak{B}$ (with respect to the standard bases), respectively. Then compute the composition $\mathrm{Alt}\,(m, f \circ g) = \mathrm{Alt}\,(m, g) \circ \mathrm{Alt}\,(m, f)$ using the basis Δ_H, $H \in \mathfrak{P}_m(\{1, \ldots, n\})$ of the K-vector space $\mathrm{Alt}\,(m, K^n)$.)

S10.71 (N o r m) Let A be a finite dimensional K-algebra. For $x \in A$, let $\lambda_x : A \to A$ be the left-multiplication $y \mapsto xy$ by x on A. Show that λ_x is a K-linear operator on A. Its determinant is called the N o r m o f x (over K) and is denoted by $\mathrm{N}_K^A(x) = \mathrm{N}(x)$.

(a) For all $x, y \in A$, $\mathrm{N}(xy) = \mathrm{N}(x)\mathrm{N}(y)$.

(b) For all $a \in K$, $\mathrm{N}(a) := \mathrm{N}(a \cdot 1_A) = a^n$, $n := \mathrm{Dim}_K A$.

(c) An element $z \in A$ is a unit in A if and only if $\mathrm{N}\,(x) \ne 0$ in K.

S10.72 For all elements z of the $\mathbb{R}$-Algebra $\mathbb{C}$, show that $\mathrm{N}_{\mathbb{R}}^{\mathbb{C}}(z) = |z|^2$. (**Hint :** See Supplement S10.71.)

S10.73 Let $A = \mathrm{M}_n(K)$ be the algebra of $n \times n$-matrices over the field K. For all $\mathfrak{A} \in A$, show that $\mathrm{N}_K^A(\mathfrak{A}) = (\mathrm{Det}\,\mathfrak{A})^n$. (**Hint :** See Supplement S10.71. — Minimal computation can be done using : $\mathrm{N}_K^A(\mathfrak{A}) = (\mathrm{Det}\,\mathfrak{A})^m$ for a fixed $m \in \mathbb{N}$. Compute this m by specialising the matrix $\mathfrak{A}$, see Corollary 9.D.9.)

S10.74 Let V be a finite dimensional $\mathbb{C}$-vector space and let $f : V \to V$ be a $\mathbb{C}$-linear operator on V. We consider V as a $\mathbb{R}$-vector space, then f is a $\mathbb{R}$-linear operator and its determinant is denoted by

$\mathrm{Det}_{\mathbb{R}} f$. Show that $\mathrm{Det}_{\mathbb{R}} f = |\mathrm{Det}\, f|^2$. In particular, if A is a finite dimensional $\mathbb{C}$-algebra, then, for all $x \in A$, show that $\mathrm{N}^A_{\mathbb{R}}(x) = |\mathrm{N}^A_{\mathbb{C}}(x)|^2$, see Supplement S10.71. (**Hint :** If $\mathfrak{A} + \mathrm{i}\,\mathfrak{B}$, $\mathfrak{A}, \mathfrak{B} \in \mathrm{M}_n(\mathbb{R})$, is the matrix of f with respect to the $\mathbb{C}$-Basis $v_1, \ldots, v_n$ of V, then

$$\begin{pmatrix} \mathfrak{A} & -\mathfrak{B} \\ \mathfrak{B} & \mathfrak{A} \end{pmatrix} \in \mathrm{M}_{2n}(\mathbb{R})$$

is the matrix of f with respect to the $\mathbb{R}$-Basis $v_1, \ldots, v_n, \mathrm{i}v_1, \ldots, \mathrm{i}v_n$ and

$$\begin{vmatrix} \mathfrak{A} & -\mathfrak{B} \\ \mathfrak{B} & \mathfrak{A} \end{vmatrix} = \begin{vmatrix} \mathfrak{A} - \mathrm{i}\,\mathfrak{B} & -\mathfrak{B} \\ \mathfrak{B} + \mathrm{i}\,\mathfrak{A} & \mathfrak{A} \end{vmatrix} = \begin{vmatrix} \mathfrak{A} - \mathrm{i}\,\mathfrak{B} & -\mathfrak{B} \\ 0 & \mathfrak{A} + \mathrm{i}\,\mathfrak{B} \end{vmatrix} .)$$

S10.75 Determine which of the following affinities of an n-dimensional oriented real affine spaces are orientation preserving: **(a)** point-reflections. **(b)** reflections of a hyperplanes along a lines and product of such r reflections, $r \in \mathbb{N}$. **(c)** transvections. **(d)** dilatations. **(e)** magnifications.

S10.76 Let E be an oriented n-dimensional $\mathbb{R}$-affine space. Suppose that the affine basis $P_0, \ldots, P_n$ represents the orientation of E. For a permutation σ in $\mathfrak{S}(\{0, \ldots, n\})$, show that the affine basis $P_{\sigma(0)}, \ldots, P_{\sigma(n)}$ represents the orientation of E if and only if σ is even. Further, show that the affine Basis $P_n, \ldots, P_0$ also represents the orientation of E if and only if $n \equiv 0$ or $n \equiv 3$ modulo 4. (**Hint :** See also Exercise 10.9 (a).)

S10.77 In every subgroup of the affine group $\mathrm{A}(E)$ of an oriented finite dimensional real affine space E which has at least one orientation reversing map, the subset of all orientation preserving maps form a subgroup of index 2.

S10.78 Suppose that the finite dimensional $\mathbb{R}$-vector space V is the direct sum of the subspaces U and W. By the following specifications of orientations on two of the spaces U, V, W a orientation on the third is determined : Suppose that $\mathfrak{u} = (u_1, \ldots, u_r)$ respectively $\mathfrak{w} = (w_1, \ldots, w_s)$ are bases of U respectively W. Then the basis $(u_1, \ldots, u_r, w_1, \ldots, w_s)$ represents the orientation of $V = U \oplus W$ if and only if the bases $\mathfrak{u}$ respectively $\mathfrak{w}$ both represents (or both don't represent) the orientations of U and W respectively. (**Hint :** Note the dependence on the sequence U and W.)

S10.79 Let V be a finite dimensional $\mathbb{R}$-vector space, $V' \subseteq V$ be a subspace of V and $\overline{V} = V/V'$ be the quotient space of V modulo V'. By the specifications of the orientations on the two of the spaces V', V, $\overline{V}$ a orientation on the third is determined : Suppose that $v'_1, \ldots, v'_r \in V'$ is a basis of V' and that the residue-classes of $v_1, \ldots, v_s \in V$ form a basis of $\overline{V}$. Show that the basis $v'_1, \ldots, v'_r, v_1, \ldots, v_s$ of V represents the orientation of V if and only if the bases $v'_1, \ldots, v'_r$ of V' and $\overline{v}_1, \ldots, \overline{v}_s$ of $\overline{V}$ both represent (or both don't represent) the orientations of V' and $\overline{V}$ respectively.

S10.80 Determine which of the following bases of $\mathbb{R}^n$ represent the standard orientation :

(a) $n = 2$; $v_1 = (1, 1)$, $v_2 = (1, -1)$.

(b) $n = 3$; $v_1 = (-1, 0, 1)$, $v_2 = (0, -1, 1)$, $v_3 = (1, -1, 1)$.

(c) $n = 4$; $v_1 = (1, 1, 1, 1)$, $v_2 = (1, 2, 1, 1)$, $v_3 = (1, 1, 3, 1)$, $v_4 = (1, 1, 1, 4)$.

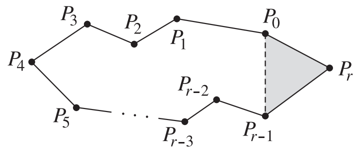
S10.81 **(a)** Every $\mathbb{C}$-linear isomorphism of finite dimensional complex vector spaces is orientation preserving. (see Example 9.F.6.)

(b) A $\mathbb{C}$-anti-linear isomorphism of finite dimensional complex vector spaces (see Example 5.C.7.) is orientation preserving if and only if their common complex dimension is even.

S10.82 Let E be a real affine plane with the volume-function $\lambda_{\mathfrak{v}}$ with respect to the basis v_1, v_2 of the space of the translations of E and $P_0, \ldots, P_r$, $r \geq 2$, be points with the coordinates (a_j, b_j), $j = 0, \ldots, r$, with respect to an affine coordinate system $O; v_1, v_2$. Furthermore, let $[P_0, P_1, \ldots, P_r, P_0]$ be a simple closed polygon, i. e. the edges meet exactly at the adjacent vertices. Show that the surface area of enclosed polygon is, up to a sign, equal to

$$\frac{1}{2}\left(\mathrm{Det}\begin{pmatrix} a_0 & a_1 \\ b_0 & b_1 \end{pmatrix} + \cdots + \mathrm{Det}\begin{pmatrix} a_{r-1} & a_r \\ b_{r-1} & b_r \end{pmatrix} + \mathrm{Det}\begin{pmatrix} a_r & a_0 \\ b_r & b_0 \end{pmatrix}\right).$$

(**Remark :** What do we mean by sign? Think about the orientation of E. — For the inductive-step from $r-1$ to r use: by suitable numbering of the vertices of the polygon with vertices $P_0,\ldots,P_{r-1}$ and the complement of the triangle with the vertices P_{r-1}, P_r, P_0 with only one common edge $[P_{r-1}, P_0]$.)



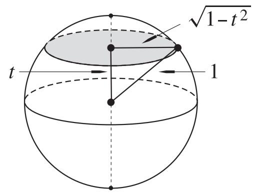
S10.83 The volume of the ellipsoid

$$\Big\{(x_1,\ldots,x_n)\in\mathbb{R}^n \;\Big|\; \frac{x_1^2}{a_1^2}+\cdots+\frac{x_n^2}{a_n^2}\le 1\Big\}\subseteq\mathbb{R}^n,$$

$a_i\in\mathbb{R}_+^\times$, $1\le i\le n$, is $\omega_n a_1\cdots a_n$, where ω_n is the volume of the unit-sphere

$$\big\{(x_1,\ldots,x_n)\in\mathbb{R}^n \;\big|\; x_1^2+\cdots+x_n^2\le 1\big\}.$$

(**Remarks :** Note that $\omega_n=\pi^{n/2}/(n/2)!$; this needs a proof and uses Measure Theory. The volume of the unit-sphere is $\omega_n=\pi^{n/2}/(n/2)!$. — To compute the volume [9] of the unit-ball $\overline{B}^{\,n}:=\overline{\mathrm{B}}(0;1)=\{x\in\mathbb{R}^n \mid \|x\|\le 1\}$ in $\mathbb{R}^n$, where $\|-\|$ denote the standard Euclidean norm.



We put $\omega_n:=\lambda^n(\overline{B}^{\,n})$. The volume of a ball with radius r is then $\omega_n r^n$. (Why?) It is easy to check that $\omega_0=1$, $\omega_1=2$, $\omega_2=\pi$ and the equality of A r c h i m e d e s : $\omega_3=\frac{4}{3}\pi$, since the surface-area $\lambda^2\big((\{t\}\times\mathbb{R}^2)\cap\overline{B}^{\,3}\big)=\pi(1-t^2)$, $-1\le t\le 1$, is a polynomial of degree 2 (≤ 3) in t.)

S10.84 Sketch the picture of the set $M:=H_1\cap H_2\cap H_3$ in $\mathbb{R}^2$, where

$$H_i:=\big\{(x,y)\in\mathbb{R}^2 \;\big|\; f_i(x,y)\ge 0\big\},$$

$i=1,2,3$, and $f_1(x,y):=x+3y+1$, $f_2(x,y):=-5x+y+1$, $f_3(x,y):=x-y+3$ and compute its area.

S10.85 Let $f_1,\ldots,f_n$ be a basis of the space of linear forms on $\mathbb{R}^n$. Let $\mathfrak{A}:=(a_{ij})\in\mathrm{GL}_n(\mathbb{R})$ be the transition matrix from the dual basis $e_1^*,\ldots,e_n^*$ (with respect to the standard basis $e_1,\ldots,e_n$ of $\mathbb{R}^n$) to the basis $f_1,\ldots,f_n$. Therefore $f_j=\sum_{i=1}^n a_{ij}e_i^*$, and $f_1,\ldots,f_n$ is the dual basis with respect to the basis $v_j=\sum_{i=1}^n b_{ij}e_i$, $j=1,\ldots,n$, where $\mathfrak{B}:=(b_{ij})={}^t\mathfrak{A}^{-1}$ is the contra-gradient matrix of $\mathfrak{A}$ (see Supplement S9.23). Let $d:=|\mathrm{Det}\,\mathfrak{A}|$. Show that

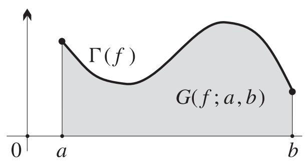
(a) For $c_1,\ldots,c_n\ge 0$, the volume of $\big\{x\in\mathbb{R}^n \;\big|\; |f_i(x)|\le c_i,\ i=1,\ldots,n\big\}$ is equal to $2^n c_1\cdots c_n/d$.

(b) For $c\ge 0$, the volume of $\big\{x\in\mathbb{R}^n \;\big|\; \sum_{i=1}^n |f_i(x)|\le c\big\}$ is equal to $2^n c^n/n!\,d$.

(c) For $c\ge 0$, the volume of the ellipsoid $\big\{x\in\mathbb{R}^n \;\big|\; \sum_{i=1}^n |f_i(x)|^2\le c^2\big\}$ is equal to $\omega_n c^n/d$, where ω_n have the same meaning as in Supplement S10.83.

---

[9] In general it is difficult to compute the (volume =) Borel-Lebesgue measure $\lambda^n(M)$ of an arbitrary Borel-set $M\subseteq\mathbb{R}^n$. For subsets in $\mathbb{R}^2$, we have used the *Fundamental Theorem of Differential–and Integral Calculus*:
Theorem (F u n d a m e n t a l T h e o r e m o f D i f f e r e n t i a l – a n d I n t e g r a l C a l c u l u s) *Let $f:[a,b]\to\mathbb{R}$, $a\le b$, be a continuous function with $f\ge 0$. Then the integral $\int_a^b f(t)\,dt$ is the area of the compact set $G(f;a,b):=\{(x,y)\mid a\le x\le b,\ 0\le y\le f(x)\}$.*



(d) For $c_0, c_1, \ldots, c_n \in \mathbb{R}$ with $c_0 \le c_1 + \cdots + c_n$, the volume of the simplex

$$\{x \in \mathbb{R}^n \mid f_i(x) \le c_i\,,\ i = 1, \ldots, n\,,\ f_1(x) + \cdots + f_n(x) \ge c_0\}$$

is equal to $b^n/n!\,d$ mit $b := c_1 + \cdots + c_n - c_0$.

(**Proof:** The matrix of the linear map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ with $f(x_1, \ldots, x_n) = \big(f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)\big)$ with respect to the standard basis is the transpose ${}^t\mathfrak{A}$. Therefore $\operatorname{Det} f = \operatorname{Det} {}^t\mathfrak{A} = \operatorname{Det} \mathfrak{A} = d$ and so $|\operatorname{Det} f^{-1}| = d^{-1}$. Now by Theorem 9.G.2 and the remarks after that $\lambda^n\big(f^{-1}(M)\big) = \lambda^n(M)/d$. for every set M for which $\lambda^n(M)$ is defined.

(a) The volume of the cuboid $Q := [-c_1, c_2] \times \cdots \times [-c_n, c_n]$ is equal to the product $(2c_1) \cdots (2c_n) = 2^n c_1 \cdots c_n$ of the lengths of its sides, and it follows that $\lambda^n(Q) = \lambda^n\big(\{x \in \mathbb{R}^n \mid |f_1(x)| \le c_1, \ldots, |f_n(x)| \le c_n\}\big) = \lambda^n\big(f^{-1}([-c_1, c_2] \times \cdots \times [-c_n, c_n])\big) = 2^n c_1 \cdots c_n/d$.

(b) Since the volume of the simplex $\{y = (y_1, \ldots, y_n) \in \mathbb{R}_+^n \mid y_1 + \cdots + y_n \le c\}$ (by 9.G.4) is equal to $c^n/n!$, the volume of $M := \{y = (y_1, \ldots, y_n) \in \mathbb{R}^n \mid |y_1| + \cdots + |y_n| \le c\}$ is $2^n c^n/n!$. It follows that $\lambda^n(M) = \lambda^n\big(\{x \in \mathbb{R}^n \mid |f_1(x)| + \cdots + |f_n(x)| \le c\}\big) = \lambda^n\big(f^{-1}(M)\big) = 2^n c^n/dn!$.)

S10.86 Let $P_0, \ldots, P_n \in \mathbb{R}^n$ be affinely independent points and let S be the (convex) simplex with these vertices. Further, let $y_0, \ldots, y_n \in \mathbb{R}_+$ and H be the affine hyperplane in $\mathbb{R}^{n+1}$ through the points $(P_0, y_0), \ldots, (P_n, y_n) \in \mathbb{R}^{n+1}$. Therefore H is the graph of the affine function $h\colon \mathbb{R}^n \to \mathbb{R}$ with $h(P_i) = y_i$, $i = 0, \ldots, n$. If $T \subseteq \mathbb{R}^{n+1}$ is the solid-body in between S and H, i. e.,

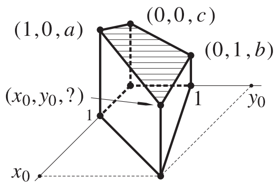
$$T := \{(x, y) \in \mathbb{R}^{n+1} \mid x \in S\,,\ 0 \le y \le h(x)\}\,,$$

then

$$\lambda^{n+1}(T) = \frac{y_0 + \cdots + y_n}{n+1}\, \lambda^n(S)\,.$$

(**Hint:** $\lambda^{n+1}(T)$ is additive in $(y_0, \ldots, y_n)$ and does not change if the values $y_0, \ldots, y_n$ are permutated. One can also assume that all y_i are equal or that all y_i other than a value y_{i_0} vanish.)

Compute the volume of the following solid-bodies in $\mathbb{R}^3$, where the top surface area is:



S10.87 The group $\mathrm{GL}_n(\mathbb{R})$, $n \in \mathbb{N}^*$, is the direct product of the groups $\mathrm{I}_n(\mathbb{R})$ of volume preserving (or u n i m o d u l a r) matrices $\mathfrak{B} \in \mathrm{GL}_n(\mathbb{R})$ with $|\operatorname{Det}\mathfrak{B}| = 1$ and the group $\mathbb{R}_+^\times \mathfrak{E}_n \cong \mathbb{R}_+^\times$ of the scalar matrices $a\mathfrak{E}_n$, $a \in \mathbb{R}_+^\times$, i. e. every matrix $\mathfrak{A} \in \mathrm{GL}_n(\mathbb{R})$ has a representation $\mathfrak{A} = a\mathfrak{B} = \mathfrak{B}a$ with uniquely determined (by $\mathfrak{A}$) elements $a \in \mathbb{R}_+^\times$ and $\mathfrak{B} \in \mathrm{I}_n(\mathbb{R})$. (**Remark:** Deduce that: Every linear automorphism f of $\mathbb{R}^n$ is the composition of a volume-preserving automorphism g and a homothecy $a \cdot \mathrm{id}$ with positive stretching-factor a, where g and $a = |\operatorname{Det} f|^{1/n}$ are uniquely determined by f. The automorphism g is called the v o l u m e - r e s e r v i n g p a r t and the scalar a is called the s t r e t c h i n g - f a c t o r of f.)