Monoid Rings

Let A be a commutative ring, and I be any set. The direct sum $A^{(I)}$ is a free A-module with the basis $\{e_i \mid i \in I\}$. The ~~t~~ element e_i is defined as

$$e_i(j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Any element ~~in~~ $a \in A^{(I)}$ can be uniquely written as

$$a = \sum_{i \in I} a_i e_i, \quad (a_i) \in A^{(I)}$$

The j^{th} component of a, $a_j = \left(\sum_{i \in I} (a_i e_i) \right)(j)$.

Consider a commutative ring A and a monoid M. We can construct a new ring $A[M]$ called the monoid ring M over A. Note $A^{(M)}$ is not a ring with the usual component-wise addition and multiplication. $A^{(M)}$ is a free module with basis $\{e_\sigma\}_{\sigma \in M}$,

$e_\sigma = (\delta_{\sigma\tau})_{\tau \in M}$. $(A^{(M)}, +)$ is a abelian group (with ~~basis~~ component-wise addition). The multiplication on $A^{(M)}$ is defined as follows:

$$\left(\sum_{\sigma \in M} a_\sigma e_\sigma \right) \left(\sum_{\tau \in M} b_\tau e_\tau \right) = \sum_{\sigma, \tau \in M} a_\sigma b_\tau e_\sigma e_\tau$$

$$= \sum_{\sigma, \tau \in M} a_\sigma b_\tau e_{\sigma \cdot \tau}$$

Note $e_\sigma \cdot e_\tau = e_{\sigma \cdot \tau}$ is well defined because $\sigma, \tau \in M$. The multiplication defined above is associative.

$$(e_\sigma \cdot e_\tau) \cdot e_\rho = e_\sigma \cdot (e_\tau \cdot e_\rho)$$

because $e_{\sigma\tau} \cdot e_\rho = e_{(\sigma\tau)\rho} = e_{\sigma(\tau\rho)} = e_\sigma e_{\tau\rho} = e_\sigma \cdot (e_{\tau\rho})$
$= e_\sigma \cdot (e_\tau \cdot e_\rho)$

The identity element is e_{i_M}, i_M is the identity in M because $e_{i_M} \cdot e_\sigma = e_\sigma \cdot e_{i_M} = e_\sigma \ \forall \sigma \in M$.
Therefore $A[M]$ is a monoid ~~ring~~ with respect to the multiplication. <u>$A[M]$ is a ring.</u> If $M = G$, where G is a group, then $A[G]$ is called a group ring G over A.

<u>Properties</u>:

(1) There exists a monoid homomorphism

$$\begin{aligned} M &\longrightarrow A[M] \\ \sigma &\longmapsto e_\sigma . \end{aligned}$$

Due to uniqueness of e_σ, this map is injective $\Rightarrow$
$\Rightarrow$ $A[M]$ is a submonoid of M.

(2) There exists a ring homomorphism

$$\begin{aligned} A &\longrightarrow A[M] \\ a &\longmapsto a \cdot e_{i_M} \end{aligned}$$

This map is injective due to uniqueness. It is easy to see that (i) $(a+b) \cdot e_{i_M} = a \cdot e_{i_M} + b \cdot e_{i_M}$,
(ii) $(a \cdot b) \cdot e_{i_M} = (a \cdot e_{i_M}) \cdot (b \cdot e_{i_M})$ #

Verify the following
(i) $A[M]$ is a A-algebra
(ii) $A[M]$ is commutative iff M is commutative.

Given any $f \in A[M]$, f can be uniquely expressed as

$$f = \sum_{\sigma \in M} a_\sigma e_\sigma, \quad (a_\sigma), e_\sigma \in A^{(M)}.$$

These $\{a_\sigma\}$, $\sigma \in M$ are called the coefficients of f.

Universal Property of $A[M]$

Let B be an A-algebra and $\alpha: M \to B$ be a monoid homomorphism. Then there exists a unique-ring homomorphism $\psi: A[M] \longrightarrow B$ such that $e_\sigma \longmapsto \alpha(\sigma)$

<u>Proof:</u> Since $A[M]$ is a free module, ~~iff~~ a ring homomorphism ψ is uniquely determined by its mapping of the basis elements e_σ, $\sigma \in M$. We only have to check ψ is a ring homomorphism. Let $f, g \in A[M]$

Then $f = \sum_\sigma a_\sigma e_\sigma$, $g = \sum_\tau b_\tau e_\tau$

$$fg = \sum_{\sigma\tau} a_\sigma b_\tau \underbrace{e_\sigma \cdot e_\tau}_{e_{\sigma\tau}} \Rightarrow \psi(f g) = \sum_{\sigma,\tau} a_\sigma b_\tau \, \alpha(\sigma\tau)$$

Since α is a monoid homomorphism

$$\psi(fg) = \sum_{\sigma,\tau} a_\sigma b_\tau \, \alpha(\sigma)\alpha(\tau)$$

$$= \left(\sum_\sigma a_\sigma \alpha(\sigma)\right)\left(\sum_\tau b_\tau \alpha(\tau)\right) = \psi(f)\psi(g)$$

$$\Longrightarrow \psi(fg) = \psi(f)\psi(g)$$

B is A-algebra $\Rightarrow$ there exists a ~~ax~~ structure ring homomorphism $\phi: A \to B$. Verify that the following diagram is commutative

$$\begin{tikzcd} A[M] \arrow[r, "\psi"] & B \\ & A \arrow[ul, "i"] \arrow[u, "\phi"'] \end{tikzcd}$$

$i: A \longrightarrow A[M]$

$a \longmapsto a e_{1_M}$

$\psi \circ i = \phi$

Examples: (1) $B = A$. Then ϕ is the identity map. α is the trivial homomorphism $M \to A = B$
$\sigma \mapsto 1$

From the universal property, there exists a ring homomorphism $\pi : A[M] \longrightarrow B = A$

$$e_\sigma \longmapsto 1$$
$$a e_{i_M} \longmapsto a$$

$$f = \sum_{\sigma \in M} a_\sigma e_\sigma \longmapsto \sum_{\sigma \in M} a_\sigma \in A$$

π is surjective. However, $\mathrm{Ker}(\pi) \neq 0 \Rightarrow \pi$ is not injective.
This is obvious because $a_\sigma e_\sigma - a_\tau e_\tau \neq 0$, but
$\pi(a_\sigma e_\sigma - a_\tau e_\tau) = 0$ if $a_\sigma = a_\tau$.

(2) $M = (\mathbb{N}, +)$ identity $i_M = 0$. Any element $f \in A[M]$ can be written as $f = \sum_{\sigma \in \mathbb{N}} a_\sigma e_\sigma$

$$e_\sigma \cdot e_\tau = e_{\sigma+\tau} \Rightarrow e_1 . e_1 = e_2$$
$$\Rightarrow e_i = \underbrace{e_1 \cdot e_1 \cdot e_1 \cdots e_i}_{i \text{ times}}$$

Therefore, $f = \sum_{i=0}^{m} a_i e_i = \sum_{i=0}^{m} a_i e_1^i = \sum_{i=0}^{m} a_i X^i$ where $X = e_1$.

(3) Suppose B is a A-algebra, then $\exists$ a structure ring homomorphism $\phi : A \to B$. Let M, N be monoids and $\beta : M \to N$ be a monoid homomorphism. $B[N]$ is an monoid ring over B.

$$M \xrightarrow{\beta} N \longrightarrow (B[N], \cdot)$$
$$\sigma \longmapsto \beta(\sigma) \longmapsto e_{\beta(\sigma)} \Rightarrow \alpha : M \longrightarrow (B[N], \cdot)$$
$$\sigma \longmapsto e_{\beta(\sigma)}$$

α is a monoid homomorphism

Note that $B[N]$ is also a A-algebra.

From the universal property, the following diagram is commutative.

$$\begin{tikzcd} a\ A \arrow[r, "\phi"] \arrow[d] & B \arrow[r, "i"] & B[N] \\ A[M] \arrow[urr] & & \end{tikzcd}$$

$e_{\beta(\sigma)}$, e_σ

(4) Let I be ~~ar~~ any set, consider the monoid $(\mathbb{N}^{(I)}, +)$, and the ring A (which is commutative). Any element $f \in A[\mathbb{N}^{(I)}]$ can be written as

$$f = \sum_{\nu \in \mathbb{N}^{(I)}} a_\nu e_\nu .$$

The I-fold direct sum $(\mathbb{N}^{(I)}, +)$ has the generators $\{\xi_i\}_{i \in I}$ defined as

$$\xi_i = (\delta_{ij})_{j \in I}$$

Therefore, any $\nu \in \mathbb{N}^{(I)}$ can be uniquely expressed as

$$\nu = \sum_{i \in I} \nu_i \xi_i .$$

Note that among all (ν_i) all but finitely many ν_i's are zero.

Using the above representation,

$$\nu \in \mathbb{N}^{(I)} \longrightarrow e_\nu = e_{\left(\sum_{i \in I} \nu_i \xi_i\right)}$$

However, the multiplication defined in $A[\mathbb{N}^{(I)}]$ implies $e_\nu \cdot e_\mu = e_{\nu+\mu} \Rightarrow e_\nu = e_{\left(\sum_{i \in I} \nu_i \xi_i\right)} = \prod_{i \in I} e_{\nu_i \xi_i}$

Since $\{\nu_i\}$ are natural numbers,

$$e_{\nu_i \xi_i} = e_{\xi_i}^{\nu_i} = \underbrace{e_{\xi_i} e_{\xi_i} e_{\xi_i} \cdots e_{\xi_i}}_{\nu_i \text{ times}}$$

$$e_\nu = e_{\left(\sum_{i\in I} \nu_i \xi_i\right)} = \prod_{i\in I} \left(e_{\xi_i}\right)^{\nu_i}$$

<u>Illustrative example:</u> Let $I = \mathbb{N}$

$$\xi_0 = (1,0,0\ldots) \longrightarrow e_{\xi_0} = e_{(1,0,0\ldots)}$$

$$2\xi_0 = (2,0,0\ldots) \longrightarrow e_{2\xi_0} = e_{(2,0,0,\ldots)}$$

$$\xi_1 = (0,1,0\ldots) \longrightarrow e_{\xi_1} = e_{(0,1,0\ldots)}$$

$$2\xi_1 = (0,2,0\ldots) \longrightarrow e_{2\xi_1} = e_{(0,2,0\ldots)}$$

Denote $X_i = e_{\xi_i}$. Let $e_\nu = e_{(\nu_0,\nu_1,\nu_2,\ldots \nu_m,0,0\ldots)}$

$$e_\nu = e_{(\nu_0,0,0\ldots)}\, e_{(0,\nu_1,0,0\ldots)}\, e_{(0,0\nu_2,0\ldots0)} \cdots e_{(0,0,\ldots\nu_m,0,0)}$$

$$= X_0^{\nu_0} X_1^{\nu_1} \cdots X_m^{\nu_m}$$

Generally,

$$\prod_{i\in I}\left(e_{\xi_i}\right)^{\nu_i} = \prod_{i\in I} X_i^{\nu_i} = X^\nu$$

Therefore, any $f \in A[\mathbb{N}^{(I)}]$ can be written as

$$f = \sum_{\nu\in\mathbb{N}^{(I)}} a_\nu e_\nu = \sum_{\nu\in\mathbb{N}^{(I)}} a_\nu \left(\prod_{i\in I} X_i^{\nu_i}\right) = \sum_{\nu\in\mathbb{N}^{(I)}} a_\nu X^\nu$$

Coming back to Example 3: Let $B = A[\mathbb{N}]$, $N = \mathbb{N}$, $M = \mathbb{N}^2$. Then $(A[\mathbb{N}])[\mathbb{N}] \cong A[\mathbb{N}^2]$

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A[\mathbb{N}] \arrow[r] & (A[\mathbb{N}])[\mathbb{N}] \\
A[\mathbb{N}^2] \arrow[urr] & &
\end{tikzcd}$$