Algebra, Arithmetic and Geometry – With a View Toward Applications / 2005

Supplementary Lectures: Friday 18:15–19:15 ; LH-1, Department of Mathematics

R1. Ordered fields

Julius Wihelm Richard Dedekind†
(1831-1916)

Unless otherwise speicified, in the following exercises the elements are in an ordered field K, for example, in the field $\mathbb{Q}$ of rational numbers, or in the field $\mathbb{R}$ of real numbers.

R1.1. For which x resp. x, y the following inequalities are true.

a). $1/|x-2| > 1/(1+|x-1|),\ x \neq 2.$ **b).** $(2-|x-1|)/|x-4| \geq 1/2,\ x \neq 4.$

c). $(|x|-1)/(x^2-1) \geq 1/2,\ x \neq \pm 1.$ **d).** $2(x+y)^2 \leq y(3x+2y)^2.$

R1.2. 1). a). If $x \geq 0$, then $x \geq (3x/(3+x))^2$. **b).** If $x \geq 1$, then $x \geq ((3x+1)/(3+x))^2$.

2). For all $m, n \in \mathbb{N}$ we have: **a).** If $0 \leq x < y$ and $n > 0$, then $0 \leq x^n < y^n$.

b). If $1 \leq x$ and $m \leq n$, then $x^m \leq x^n$. **c).** If $0 \leq x \leq 1$ and $m \leq n$, then $x^m \geq x^n$.

3). a). $(x/y) + (y/x) \geq 2$, if $x, y > 0$. **b).** $2xy \leq (x+y)^2/2 \leq x^2 + y^2$.

c). $xy + xz + yz \leq x^2 + y^2 + z^2$. **d).** $(x+y)(y+z)(z+x) \geq 8xyz$, if $x, y, z \geq 0$.

R1.3. 1). For all $n \in \mathbb{N}$, show that $(1+(x/y))^n + (1+(y/x))^n \geq 2^{n+1}$, if $x, y > 0$.

2). For all $n \in \mathbb{N}^*$, show that $((x+y)/2)^n \leq (x^n+y^n)/2 \leq (x+y)^n/2$, if $x, y \geq 0$.

3). For all x, y with $x + y \geq 0$, show that:

a). $x^3 + y^3 \geq xy(x+y)$. **b).** $(x/y^2) + (y/x^2) \geq 1/x + 1/y$, if $x, y \neq 0$.

4). For all x, y, z with $x+y \geq 0,\ x+z \geq 0,\ y+z \geq 0$, show that:

$$x^3 + y^3 + z^3 \geq \frac{1}{3}(x^2+y^2+z^2)(x+y+z).$$

R1.4. 1). Show that $\mathrm{Max}\,(x,y) = \frac{1}{2}\big(x+y+|x-y|\big)$ and $\mathrm{Min}\,(x,y) = \frac{1}{2}\big(x+y-|x-y|\big)$.

2). Let $n \in \mathbb{N}^*$. For all $x_1, \ldots, x_n,\ y_1, \ldots, y_n$ with $y_1, \ldots, y_n > 0$, show that:

$$\mathrm{Min}\,\left(\frac{x_1}{y_1}, \ldots, \frac{x_n}{y_n}\right) \leq \frac{x_1 + \cdots + x_n}{y_1 + \cdots + y_n} \leq \mathrm{Max}\,\left(\frac{x_1}{y_1}, \ldots, \frac{x_n}{y_n}\right).$$

3). a). $|x| \leq |x+y| + |y|$. **b).** $|x+y|/\big(1+|x+y|\big) \leq |x|/\big(1+|x|\big) + |y|/\big(1+|y|\big)$.

c). If $|x| \leq 1$, $|y| \leq 1$, then $|x+y| \leq 1 + xy$.

4). If $x = y + z$ and $xz \leq 0$, then $x = \theta y$ with $0 \leq \theta \leq 1$. (**Remark:** This trivial assertion is often used to deduce important error estimates.)

R1.5. 1). (Bernoulli's Inequalties) If either $x_i \geq 0$ for all i, or $0 \geq x_i \geq -1$ for all i, then show that $\prod_{i=1}^{n}(1+x_i) \geq 1+x_1+\cdots+x_n$. In particular, $(1+x)^n \geq 1+nx$ for all x with $x \geq -1$ and all $n \in \mathbb{N}$.

2). a). If $0 \leq x_i \leq 1$, $i = 1, \ldots, n$, then show that $\prod_{i=1}^{n}(1-x_i) \leq 1/(1+x_1+\cdots+x_n)$. In particular, if $0 \leq x \leq 1$, then $(1-x)^n \leq 1/(1+nx)$ for all $n \in \mathbb{N}$.

b). If $x_i \geq 0$ and $\sum_{i=1}^{n} x_i < 1$, then show that $\prod_{i=1}^{n}(1+x_i) \leq 1/(1-\sum_{i=1}^{n} x_i)$.

3). If $x_i \geq 1$, then $\prod_{i=1}^{n}(1+x_i) \geq \frac{2^n}{n+1}(1+x_1+\cdots+x_n)$. In particular, $(1+x)^n \geq \frac{2^n}{n+1}(1+nx)$ for $x \geq 1$ and $n \in \mathbb{N}$.

4). For all x with $0 \leq x$ and for all $n \geq 2$, show that $(1+x)^n \geq \frac{1}{4}n^2x^2$.

5). For all x, y with $(x, y) \neq (0, 0)$ and for all positive even natural numbers n show that:

$$x^n + x^{n-1}y + \cdots + xy^{n-1} + y^n > 0.$$

6). If $x_1, \ldots, x_n > 0$, then show that $(x_1+\cdots+x_n)(1/x_1+\cdots+1/x_n) \geq n^2$. (**Hint:** see Exercise R1.8-1) below.)

R1.6. 1). For all $x_1, \ldots, x_n$ with $x_1, \ldots, x_n > 0$ and $\prod_{i=1}^{n} x_i = 1$, show that $\prod_{i=1}^{n}(1+x_i) \geq 2^n$. Moreover, the equality holds if and only if $x_1 = \cdots = x_n = 1$. (**Hint:** (For a proof of the inductive step from $n \geq 1$ to $n+1$, let x_n be the smallest and x_{n+1} the greatest among the numbers $x_1, \ldots, x_{n+1}$. Now, use the induction hypothesis for the n positive numbers $x_1, \ldots, x_{n-1},\ x_nx_{n+1}$.)

2). For all $x_1, \ldots, x_n$ with $x_1, \ldots, x_n > 0$ and $\prod_{i=1}^{n} x_i = 1$, show that $\sum_{i=1}^{n} x_i \geq n$. Moreover, the equality holds if and only if $x_1 = \cdots = x_n = 1$. (**Hint:** Use the hint given in the exercise 1) above.)

3). For all $x_1, \ldots, x_n$ with $x_1, \ldots, x_n > 0$ and $\sum_{i=1}^{n} x_i = n$, show that $\prod_{i=1}^{n} x_i \leq 1$. Moreover, the equality holds if and only if $x_1 = \cdots = x_n = 1$. (**Hint:** One can attribute this to exercise 2) above or again a similar induction proof as in exercise 1) above (the induction hypothesis is applied to $x_1, \ldots, x_{n-1}, x_n+x_{n+1}-1$).)

R1.7. Let $n \in \mathbb{N}^*$. For all $x_1, \ldots, x_n$ with $x_1, \ldots, x_n > 0$, show that

$$\left(\frac{x_1+\cdots+x_n}{n}\right)^n \geq x_1\cdots x_n \geq \left(\frac{n}{\frac{1}{x_1}+\cdots+\frac{1}{x_n}}\right)^n.$$

The equality holds if and only if $x_1 = \cdots = x_n$. (**Remark:** For positive real numbers $x_1, \ldots, x_n$,

$$a := \frac{x_1+\cdots+x_n}{n}$$

is called the arithmetic mean,

$$g := \sqrt[n]{x_1\cdots x_n}$$

is called the geometric mean [1]) and

$$h := n\big/\left(\tfrac{1}{x_1}+\cdots+\tfrac{1}{x_n}\right)$$

is called the harmonic mean of $x_1, \ldots, x_n$. Therefore we have $a \geq g \geq h$.) (**Hint:** For a proof fo the first inequality use the Exercise R1.6-2) for $x_1/a, \ldots, x_n/a$ or in the case $K = \mathbb{R}$ use exercise R1.6-3) for $x_1/g, \ldots, x_n/g$. The second inequality follows from the first. A sequence of positive real numbers is called arithmetic resp. geometric resp. harmonic, if every member of the sequence (apart from the first) is the arithmetic resp. geometric resp. harmonic mean of its both neighbour members. A sequence is arithmetic if and only if the sequence of the reciprocals is harmonic. For example, from the arithmetic sequence $1, 2, 3, \ldots$ of positive natural numbers, we get the harmonic sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots$ of the unit-fractions.)

[1]) For the existence of n-th roots in $\mathbb{R}$ see Example 4.F.9.

R1.8. 1). For all $x_1, \ldots, x_n,\ y_1, \ldots, y_n$, prove the following Cauchy-Schwarz inequality

$$\Big(\sum_{i=1}^{n} x_i y_i\Big)^2 \le \Big(\sum_{i=1}^{n} x_i^2\Big)\Big(\sum_{i=1}^{n} y_i^2\Big).$$

(**Hint:** Use $(\sum_{i=1}^n x_i^2)(\sum_{i=1}^n y_i^2) = (\sum_{i=1}^n x_i y_i)^2 + \sum_{1\le i<j\le n}(x_i y_j - x_j y_i)^2$ or in the case $K = \mathbb{R}$, use $x := \sum_{i=1}^n x_i^2 > 0,\ y := \sum_{i=1}^n y_i^2 > 0$ and add the n inequalities $x_i y_i/\sqrt{xy} \le (x_i^2/x + y_i^2/y)/2$.)

2). For all $x_1, \ldots, x_n$, prove that $\Big(\sum_{i=1}^{n} x_i\Big)^2 \le n \sum_{i=1}^{n} x_i^2$.

3). Let n be a positive natural number. Then: **a).** $\Big(\sum_{k=1}^{n} 1/k\Big)^2 < 2n$. **b).** $\Big(\sum_{k=n+1}^{2n} 1/k\Big)^2 < 1/2$.

4). (Minkowski's inequality) For all $x_1, \ldots, x_n,\ y_1, \ldots, y_n \in \mathbb{R}$, prove that

$$\sqrt{\sum_{i=1}^{n}(x_i + y_i)^2} \le \sqrt{\sum_{i=1}^{n} x_i^2} + \sqrt{\sum_{i=1}^{n} y_i^2}.$$

(**Hint:** For a proof square both sides and use the Cauchy-Schwarz inequality in the exercise 1) above.)

5). Show that the union of finitely many bounded subsets of K is again bounded.

R1.9. 1). Let $f : \mathbb{R} \to \mathbb{R}$ be the function $x \mapsto (x-1)x(x+1)$. On the number-line Sketch the pictures of the set of points $\{x \in \mathbb{R} \mid f(x) \ge 0\}$ and $\{x \in \mathbb{R} \mid f(x) \le 0\}$.

2). For the following functions $f : \mathbb{R}^2 \to \mathbb{R}$, sketch the pictures of the set of points (x, y) with $f(x, y) > 1$ resp. $= 1$ resp. < 1 in the number-plane $\mathbb{R}^2$: **a).** $f(x, y) = |x - y|$. **b).** $f(x, y) = x^2 y^2$. **c).** $f(x, y) = x^2 + xy + 1$.

3). Sketch the picture of the set $\{(x, y) \in \mathbb{R}^2 \mid x^2 \le y \le x^4\} \subseteq \mathbb{R}^2$.

4). Sketch the picture of the set of pairs $(x, y) \in \mathbb{R}^2$ with $xy > x + y$ resp. $xy = x + y$ resp. $xy < x + y$.

5). Let $k, n \in \mathbb{N}^*$, $k \le n$ and let $x_1, \ldots, x_n$ be positive number. Suppose that for every k, the product of k positive numbers among $x_1, \ldots, x_n$ is ≥ 1. Then prove that $x_1 \cdots x_n \ge 1$.

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† **Julius Wihelm Richard Dedekind (1831-1916)** was born on 6 Oct 1831 in Braunschweig, duchy of Braunschweig (now Germany) and died on 12 Feb 1916 in Braunschweig, duchy of Braunschweig (now Germany). Richard Dedekind's father was a professor at the Collegium Carolinum in Brunswick. His mother was the daughter of a professor who also worked at the Collegium Carolinum. Richard was the youngest of four children and never married. He was to live with one of his sisters, who also remained unmarried, for most of his adult life.

He attended school in Brunswick from the age of seven and at this stage mathematics was not his main interest. The school, Martino-Catharineum, was a good one and Dedekind studied science, in particular physics and chemistry. However, physics became less than satisfactory to Dedekind with what he considered an imprecise logical structure and his attention turned towards mathematics.

The Collegium Carolinum was an educational institution between a high school and a university and he entered it in 1848 at the age of 16. There he was to receive a good understanding of basic mathematics studying differential and integral calculus, analytic geometry and the foundations of analysis. He entered the University of Göttingen in the spring of 1850 with a solid grounding in mathematics.

Göttingen was a rather disappointing place to study mathematics at this time, and it had not yet become the vigorous research centre that it turned into soon afterwards. Mathematics was directed by M A Stern and G Ulrich. Gauss also taught courses in mathematics, but mostly at an elementary level. The physics department was directed by Listing and Wilhelm Weber. The two departments combined to initiate a seminar which Dedekind joined from its beginning. There he learnt number theory which was the most advanced material he studied. His other courses covered material such as the differential and integral calculus, of which he already had a good understanding. The first course to really make Dedekind enthusiastic was, rather surprisingly, a course on experimental physics taught by Weber. More likely it was Weber who inspired Dedekind rather than the topic of the course.

In the autumn term of 1850, Dedekind attended his first course given by Gauss. It was a course on least squares and ... fifty years later Dedekind remembered the lectures as the most beautiful he had ever heard, writing that he had followed Gauss with constantly increasing interest and that he could not forget the experience.

Dedekind did his doctoral work in four semesters under Gauss's supervision and submitted a thesis on the theory of Eulerian integrals. He received his doctorate from Göttingen in 1852 and he was to be the last pupil of Gauss. However he was not well trained in advanced mathematics and fully realised the deficiencies in his mathematical education.

At this time Berlin was the place where courses were given on the latest mathematical developments but Dedekind had not been able to learn such material at Göttingen. By this time Riemann was also at Göttingen and he too found that the mathematical education was aimed at students who were intending to become secondary school teachers, not those with the very top abilities who would go on to research careers. Dedekind therefore spent the two years following the award of his doctorate learning the latest mathematical developments and working for his habilitation.

In 1854 both Riemann and Dedekind were awarded their habilitation degrees within a few weeks of each other. Dedekind was then qualified as a university teacher and he began teaching at Göttingen giving courses on probability and geometry.

Gauss died in 1855 and Dirichlet was appointed to fill the vacant chair at Göttingen. This was an extremely important event for Dedekind who found working with Dirichlet extremely profitable. He attended courses by Dirichlet on the theory of numbers, on potential theory, on definite integrals, and on partial differential equations. Dedekind and Dirichlet soon became close friends and the relationship was in many ways the making of Dedekind, whose mathematical interests took a new lease of life with the discussions between the two. Bachmann, who was a student in Göttingen at this time ... recalled in later years that he only knew Dedekind by sight because Dedekind always arrived and left with Dirichlet and was completely eclipsed by him.

Dedekind wrote in a letter in July 1856 *What is most useful to me is the almost daily association with Dirichlet, with whom I am for the first time beginning to learn properly; he is always completely amiable towards me, and he tells me without beating about the bush what gaps I need to fill and at the same time he gives me the instructions and the means to do it. I thank him already for infinitely many things, and no doubt there will be many more.*

Dedekind certainly still continued to learn mathematics at this time as a student would by attending courses, such as those by Riemann on abelian functions and elliptic functions. Around this time Dedekind studied the work of Galois and he was the first to lecture on Galois theory when he taught a course on the topic at Göttingen during this period.

While at Göttingen, Dedekind applied for J L Raabe's chair at the Polytechnikum in Zürich. Dirichlet supported his application writing that Dedekind was 'an exceptional pedagogue'. In the spring of 1858 the Swiss councillor who made appointments came to Göttingen and Dedekind was quickly chosen for the post. Dedekind was appointed to the Polytechnikum in Zürich and began teaching there in the autumn of 1858.

In fact it was while he was thinking how to teach differential and integral calculus, the first time that he had taught the topic, that the idea of a Dedekind cut came to him. He recounts that the idea came to him on 24 November 1858. His idea was that every real number r divides the rational numbers into two subsets, namely those greater than r and those less than r. Dedekind's brilliant idea was to represent the real numbers by such divisions of the rationals.

Dedekind and Riemann travelled together to Berlin in September 1859 on the occasion of Riemann's election to the Berlin Academy of Sciences. In Berlin, Dedekind met Weierstrass, Kummer, Borchardt and Kronecker.

The Collegium Carolinum in Brunswick had been upgraded to the Brunswick Polytechnikum by the 1860s, and Dedekind was appointed to the Polytechnikum in 1862. With this appointment he returned to his home town and even to his old educational establishment where his father had been one of the senior administrators for many years. Dedekind remained there for the rest of his life, retiring on 1 April 1894. He lived his life as a professor in Brunswick ... in close association with his brother and sister, ignoring all possibilities of change or attainment of a higher sphere of activity. The small, familiar world in which he lived completely satisfied his demands: in it his relatives completely replaced a wife and children of his own and there he found sufficient leisure and freedom for scientific work in basic mathematical research. He did not feel pressed to have a more marked effect in the outside world: such confirmation of himself was unnecessary.

After he retired, Dedekind continued to teach the occasional course and remained in good health in his long retirement. The only spell of bad health which Dedekind had experienced was 10 years after he was appointed to the Brunswick Polytechnikum when he had a serious illness, shortly after the death of his father. However he completely recovered and, as we mentioned, remained in good health.

Dedekind made a number of highly significant contributions to mathematics and his work would change the style of mathematics into what is familiar to us today. One remarkable piece of work was his redefinition of irrational numbers in terms of Dedekind cuts which, as we mentioned above, first came to him as early as 1858. He published this in Stetigkeit und Irrationale Zahlen in 1872. In it he wrote *Now, in each case when there is a cut (A1, A2) which is not produced by any rational number, then we create a new, irrational number a, which we regard as completely defined by this cut; we will say that this number a corresponds to this cut, or that it produces this cut.*

As well as his analysis of the nature of number, his work on mathematical induction, including the definition of finite and infinite sets, and his work in number theory, particularly in algebraic number fields, is of major importance.

Dedekind loved to take his holidays in Switzerland, the Austrian Tyrol or the Black Forest in southern Germany. On one such holiday in 1874 he met Cantor while staying in the beautiful city of Interlaken and the two discussed set theory. Dedekind was sympathetic to Cantor's set theory as is illustrated by this quote from Was sind und was sollen die Zahlen (1888) regarding determining whether a given element belongs to a given set. In what way the determination comes about, or whether we know a way to decide it, is a matter of no consequence in what follows. The general laws that are to be developed do not depend on this at all. In this quote Dedekind is arguing against Kronecker's objections to the infinite and, therefore, is agreeing with Cantor's views.

Among Dedekind's other notable contributions to mathematics were his editions of the collected works of Peter Dirichlet, Carl Gauss, and Georg Riemann. Dedekind's study of Dirichlet's work did, in fact, to lead to his own study of algebraic number fields, as well as to his introduction of ideals. Dedekind edited Dirichlet's lectures on number theory and published these as Vorlesungen über Zahlentheorie in 1863. Although the book is assuredly based on Dirichlet's lectures, and although Dedekind himself referred to the book throughout his life as Dirichlet's, the book itself was entirely written by Dedekind, for the most part after Dirichlet's death.

It was in the third and fourth editions of Vorlesungen über Zahlentheorie, published in 1879 and 1894, that Dedekind wrote supplements in which he introduced the notion of an ideal which is fundamental to ring theory. Dedekind formulated his theory in the ring of integers of an algebraic number field. The general term 'ring' does not appear, it was introduced later by Hilbert.

Dedekind, in a joint paper with Heinrich Weber published in 1882, applies his theory of ideals to the theory of Riemann surfaces. This gave powerful results such as a purely algebraic proof of the Riemann-Roch theorem.

Dedekind's work was quickly accepted, partly because of the clarity with which he presented his ideas and partly since Heinrich Weber lectured to Hilbert on these topics at the University of Königsberg. Dedekind's notion of ideal was taken up and extended by Hilbert and then later by Emmy Noether. This led to the unique factorisation of integers into powers of primes to be generalised to ideals in other rings.

In 1879 Dedekind published Über die Theorie der ganzen algebraischen Zahlen which was again to have a large influence on the foundations of mathematics. In the book Dedekind ... presented a logical theory of number and of complete induction, presented his principal conception of the essence of arithmetic, and dealt with the role of the complete system of real numbers in geometry in the problem of the continuity of space. Among other things, he provides a definition independent of the concept of number for the infiniteness or finiteness of a set by using the concept of mapping and treating the recursive definition, which is so important to the theory of ordinal numbers.

Dedekind's brilliance consisted not only of the theorems and concepts that he studied but, because of his ability to formulate and express his ideas so clearly, he introduced a new style of mathematics that been a major influence on mathematicians ever since.

Dedekind's legacy ... consisted not only of important theorems, examples, and concepts, but a whole style of mathematics that has been an inspiration to each succeeding generation.

Many honours were given to Dedekind for his outstanding work, although he always remained extraordinarily modest regarding his own abilities and achievements. He was elected to the Göttingen Academy (1862), the Berlin Academy (1880), the Academy of Rome, the Leopoldino-Carolina Naturae Curiosorum Academia, and the Académie des Sciences in Paris (1900). Honorary doctorates were awarded to him by the universities of Kristiania (Oslo), Zurich and Brunswick.