Algebra, Arithmetic and Geometry – With a View Toward Applications / 2005
Supplementary Lectures : Friday 18:15–19:15 ; LH-1, Department of Mathematics

R3. Consequences of Completeness

Bernard Placidus Johann Nepomuk Bolzano†
(1781-1848)

R3.1. 1). Determine all accumulation points, limit inferior and limit superior of the sequence :

$$(-1)^n/2 + (-1)^{n(n+1)/2}/3\,.$$

2). Give an example of a sequence for which the set of all accumulation points is precisely the set of natural numbers.

R3.2. 1). Let $A \subseteq \mathbb{R}, A \neq \emptyset$, and $-A := \{-x \mid x \in A\}$. Then show that A is bounded below if and only if $-A$ is bounded above. Moreover, in this case we have $\operatorname{Inf} A = -\operatorname{Sup}(-A)$.

2). Let $A, B \subseteq \mathbb{R}, A \neq \emptyset, B \neq \emptyset$. We put

$$A + B := \{x + y \mid x \in A, y \in B\} \quad \text{and} \quad A \cdot B := \{xy \mid x \in A, y \in B\}\,.$$

a). Show that $A + B$ is bounded above (resp. below) if and only if both A and B are bounded above (resp. below). Moreover, in this case

$$\operatorname{Sup}(A + B) = \operatorname{Sup} A + \operatorname{Sup} B \quad (\text{resp. } \operatorname{Inf}(A + B) = \operatorname{Inf} A + \operatorname{Inf} B)\,.$$

b). If $A \neq \{0\} \neq B$, then show that $A \cdot B$ is bounded if and only if both A and B are bounded.

c). If A and B are bounded and if $A, B \subseteq \mathbb{R}_+$, then show that $\operatorname{Sup}(A \cdot B) = (\operatorname{Sup} A) \cdot (\operatorname{Sup} B)$.

R3.3. Prove the following theorem on Dedekind's Cuts : Let A and B be non-empty subsets of $\mathbb{R}$ with $a < b$ for all $a \in A$ and all $b \in B$, then there exists a real number x such that $a \leq x \leq b$ for all $a \in A, b \in B$. — Moreover, if $A \cup B = \mathbb{R}$, then this real number x is uniquely determined and $x = \operatorname{Sup} A = \operatorname{Inf} B$. (**Remark :** In this case the real number x defines the well-known Dedekind's Cut (A, B).)

R3.4. 1). Let $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$. Then :

a). Show that x is an accumulation point of A if and ony if every neighbourhood of x contains a point of a different from x.

b). Show that x is an accumulation point of A if and ony if there exists a sequence (x_n) in A with pairwise distinct members, i.e. $x_n \neq x_m$ for $n \neq m$ and which converges to x.

c). Show that x is a boundary point of A if and ony if there exists a sequence (x_n) in A which converges to x.

2). The set of all accumulation points of a subset A of $\mathbb{R}$ is closed.

3). Let $A \subseteq \mathbb{R}$. Then show that $\overline{A}$ is closed and $\mathring{A}$ is open in $\mathbb{R}$. (**Remark :** The subset $\overline{A}$ is called the closure of A and the subset $\mathring{A}$ is called the interior or open-ker of A.)

4). For $A, B \subseteq \mathbb{R}$, show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$, $(A \cap B)^\circ = \mathring{A} \cap \mathring{B}$, further, $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$ and $(A \cup B)^\circ \supseteq \mathring{A} \cup \mathring{B}$. Show by examples that both the last inclusions can be proper.

5). Determine $\overline{A}$ and $\mathring{A}$ for the following subsets A of $\mathbb{R}$:

a). $\{1/n \mid n \in \mathbb{N}^*\}$, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R} \setminus \mathbb{Q}$, **b).** $[a, b]$, $]a, b[$, $[a, b[$, $]a, b]$ mit $a, b \in \mathbb{R}$, $a < b$,

c). $\{a/g^n \mid a \in \mathbb{Z},\ n \in \mathbb{N}\}$ mit $g \in \mathbb{N}$, $g \geq 2$ fest.

R3.5. 1). Every sequence of real numbers has an (infinite) monotone subsequence.

2). A sequence of real numbers is convergent if and only if it is bounded and has exactly one accumulation point. (**Remark :** This proves once again the Cauchy's Convergence Criterion (using the theorem of Weierstraß-Bolzano).)

3). Let (x_n) be a bounded sequence of real numbers. Show that

$$\limsup x_n = \lim_{n\to\infty} \big(\mathrm{Sup}\{x_m \mid m \geq n\}\big) \quad \text{and} \quad \liminf x_n = \lim_{n\to\infty} \big(\mathrm{Inf}\{x_m \mid m \geq n\}\big).$$

4). Let (x_n) be a bounded sequence of real numbers. Show that:

$\limsup x_n = \mathrm{Inf}\,\{x \in \mathbb{R} \mid x \geq x_n \text{ for almost all } n\} = \mathrm{Sup}\,\{x \in \mathbb{R} \mid x \leq x_n \text{ for infinitely many } n\}$ and

$\liminf x_n = \mathrm{Sup}\,\{x \in \mathbb{R} \mid x \leq x_n \text{ for almost all } n\} = \mathrm{Inf}\,\{x \in \mathbb{R} \mid x \geq x_n \text{ for infinitely many } n\}$.

5). Let (x_n) and (y_n) bounded sequences of real numbers. Show that

a).

$$\begin{aligned}\liminf x_n + \liminf y_n \leq \liminf (x_n + y_n) &\leq \limsup x_n + \liminf y_n \\ &\leq \limsup (x_n + y_n) \leq \limsup x_n + \limsup y_n .\end{aligned}$$

b). The inequalities in the part a) are also valid for all non- negative x_n and y_n if the plus sign is replaced by the multiplication.

R3.6. 1). A subset of $\mathbb{R}$ is called *perfect* if it is equal to the set of all of its accumulation points. A perfect set is necessarily closed. Show that every non-empty perfect set is uncountable. (**Remark :** One can show that every non-empty perfect subset $A \subseteq \mathbb{R}$ has the cardinality of the continuum. One can also use a similar argument to that given in the example 4.F.8.)

2). Let $A \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is called a *condensation point* of A if every neighbourhood of x contains uncountabley many points of A.

a). Every uncountable subset A of $\mathbb{R}$ has at least one condensation point. (**Hint :** Reduced to the case when A is bounded and then complete as in 4.G.3.)

b). The set of all condensation points of A is perfect. In particular, it is uncountable if A itself is uncountable. (**Hint :** See exercise 1) above.)

c). Every closed subset of $\mathbb{R}$ is the disjoint union of a countable perfect sets. Moreover, this decomposition is unique. (**Hint :** Every point of a perfect set in $\mathbb{R}$ is a condensation point of this set.)

R3.7. 1). A subset $A \subseteq \mathbb{R}$ is an interval if and only if A contains all the numbers between any two of its elements.

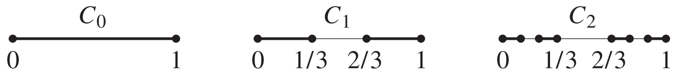
2). Let $A \subseteq \mathbb{R}$.

a). Define a relation $\sim$ on A by $a \sim b$, $a, b \in A$ if the closed interval with end points a and b is completely contained in A. Show that $\sim$ is an equivalence relation on A; its equivalence classes are intervals and are called the *connected components* of A. If all these components of A are singletons, then A is called *totally disconnected*. For example, the sets $\mathbb{R} \setminus \mathbb{Q}$ and $\mathbb{Q}$ (and hence every countable subset of $\mathbb{R}$) are totally disconnected. Every subset $A \subseteq \mathbb{R}$ has at most countably many connected components with more than one point.

b). If A is open, then A disjoint union of countably many open intervals (namely, the connected components of A).

c). If A is closed, then all connected components of A are closed intervals. (**Remark:** there might be uncountably many connected components, see the part d) below.)

d). (Cantor's Discontinuum) Let $C_0 := [0,1]$ and $C_1 := C_0 \setminus]1/3, 2/3[$. More generally, C_{n+1} is obtained from C_n by removing the open middle third from every connected component of C_n, $n \in \mathbb{N}$.



The subset $C := \bigcap_{n=0}^{\infty} C_n$ is called the Cantor's Discontinuum or the Cantor's wiping-set. Show that: (1) A number $x \in [0,1]$ belongs to C if and only if there exists a ternary expansion of x which does not contain the digit 1 (see Example 4.F.12). [1]) (2) C is a perfect (closed) totally disconnected subset of $\mathbb{R}$ and hence has cardinality of the continuum.

3). Let A be a non-empty subset of $\mathbb{R}$ which is closed as well as open. Show that $A = \mathbb{R}$. (**Hint:** Consider the connected components of A.)

R3.8. a). $\mathbb{R}$ cannot be represented as disjoint union of countably many bounded closed intervals. (**Hint:** Suppose that $\mathbb{R} = \biguplus_{n\in\mathbb{N}} [a_n, b_n]$. Then $\mathbb{R} \setminus \biguplus_{n\in\mathbb{N}}]a_n, b_n[$ is a perfect subset of $\mathbb{R}$ a contradiction to Exercise R3.6-1). — For an illustration consider the following example: Let $\overline{I}_0, \overline{I}_1, \overline{I}_2, \ldots$ be the list of all open intervals with their boundary points which were removed from the subsets C_n, $n \in \mathbb{N}$, in the construction of the Cantor's discontinuum in Exercise R3.7-2)-d). Then $\bigcup_{n\in\mathbb{N}} \overline{I}_n$ is *not* the full (open) unit interval $]0,1[$. Which points are missing ?)

b). More generally than the part a) we have: $\mathbb{R}$ cannot be represented as disjoint union of countably many closed and bounded (i.e., compact) subsets of $\mathbb{R}$. (**Hint:** One attributes this to a): Suppose that $\mathbb{R} = \biguplus_{n\in\mathbb{N}} K_n$ with bounded closed subsets K_n. We may assume that each $K_n \neq \emptyset$ and is contained in a connected component of the open subset $\mathbb{R} \setminus \bigcup_{k=0}^{n-1} K_k$, since K_n intersects with only finitely many connected components by the theorem of Weierstrass-Bolzano. Let $a_n := \operatorname{Inf} K_n$ and $b_n := \operatorname{Sup} K_n$. Now, we recursively construct a closed bounded intervals I_n in the following way: $I_0 = [a_0, b_0]$; $I_n := I_{n-1}$, if $K_n \subseteq \bigcup_{k=0}^{n-1} I_k$, resp. $I_n := [a_n, b_n]$ otherwise. Then $\mathbb{R}$ is the disjoint union of the distinct intervals in the sequence $I_0, I_1, I_2, \ldots$.) (**Remark:** An appropriate corresponding statement also holds for $\mathbb{R}^m$, $m \geq 2$. A decomposition of $\mathbb{R}^m$ into closed bounded subsets induces a similar decomposition of every line in $\mathbb{R}^m$. For further generalisation see ????. Since the times of Zenon of Elea, the result of the present problem is readily used as an argument against atomism. If you want to describe the continuum in an atomistic way, you have necessarily to admit uncountable sets. This is one of the great discoveries of Cantor. In 1884 he writes (in a letter to Mittag-Leffler): "I believe that the entirety of the body atoms is of the first cardinality, whereas the entirety of the Aether atoms is of the second cardinality". — By first cardinality Cantor denotes the cardinality $\aleph_0$ of the natural numers and by $\aleph_1$ the smallest uncountable cardinality. That $\aleph_1$ is the cardinality of the real numbers, is the so called continuum hypothesis, which was not proved by Cantor and — as we know today — can neither be disproved nor be proved in the context of the usual axiomatic set theory (Theorem of Gödel resp. Theorem of Cohen).)

R3.9. a). Let $x = a/b$ with relatively prime integers a, b, $b > 0$. Show that the sequence $nx - [nx]$, $n \in \mathbb{N}$ has exactly b accumulation points $0, 1/b, \ldots, (b-1)/b$.

b). Let $x \in \mathbb{R}$ be irrational. Show that the set of all accumulation points of the sequence $x_n := nx - [nx]$, $n \in \mathbb{N}$, is the closed unit interval $[0,1]$. (**Hint:** Use the following steps: (1) (x_n) has an accumulation point in $[0,1]$. (2) Either 0 or 1 is an accumulation point of (x_n). (3) Every point of $[0,1]$ is an accumulation point of (x_n). – Using the continued fractions of x, see Example 4.F.13, one can give a constructive proof of this exercise. For $\varphi = (1+\sqrt{5})/2$ in R2.3-7) give as small as possible $n \in \mathbb{N}$ with $|n\varphi - [n\varphi] - \frac{1}{2}| \leq 10^{-6}$.)

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[1]) Therefore starting from a place all digits of x in a ternary expansion are equal to 2, e.g. $1/3 = (0,1)_3 = (0,0222\ldots)_3 \in C$.

R3.10. In the power set $\mathfrak{P}(\mathbb{N})$ of $\mathbb{N}$ there are uncountable chains and hence there are uncountable subsets which are totally ordered with respect to the natural inclusion. (**Remark :** (B. Kaup — This is surprisingly simple perhaps surprising and in any case to prove, remember that $\mathbb{Q}$ is also countable.) Moreover, there are uncountable subsets in $\mathfrak{P}(\mathbb{N})$ whose elements are *almost disjoint*, i.e. their intersection is finite.

(**Remark :** In this connection we also mention the construction of the real numbers due to R. Dedekind, which also solves the first part of the above problem and the ideas of the Dedekind cuts follows, see Exercise R3.3. Each real number α determines the well-known Dedekind's cut $A_\alpha := \{x \in \mathbb{Q} \mid x < \alpha\}$ in $\mathbb{Q}$, where by a Dedekind's cut (in $\mathbb{Q}$) we mean a non-empty, bounded above subset of $\mathbb{Q}$ without the greatest element and which contains with each element all smaller elements of $\mathbb{Q}$. On the other hand each such a cut A in $\mathbb{Q}$ the cut corresponding to a real number, namely $A = A_{\mathrm{Sup}\,A}$, see 4.G.9. The set $\mathbb{R}$ of the real numbers can therefore be identified with the set of the Dedekind cuts in $\mathbb{Q}$. One can define thus following Dedekind turned around the set $\mathbb{R}$ as the set of the Dedekind cuts, which is possible to describe alone by $\mathbb{Q}$. Then $\mathbb{R} \subseteq \mathfrak{P}(\mathbb{Q})$ and the order on $\mathbb{R}$ is induced by the natural inclusion from $\mathfrak{P}(\mathbb{Q})$. Theorem 4.G.9 on the existence of the least upper bound Sup $\mathfrak{A}$ for a non-empty bounded above subset $\mathfrak{A} \subseteq \mathbb{R}$ is then clear : $\mathrm{Sup}\,\mathfrak{A} = \bigcup_{A\in\mathfrak{A}} A$, in particular, also the validity of the completenesss axiom 4.F.2 of Carathéodory : If $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$ is a monotone increasing sequence in $\mathbb{R}$ which is bounded above, then $\lim_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} A_n$. (These "simple" proofs become possible since Cantor had prepared to define infinite unions of sets.) The addition in $\mathbb{R}$ is simply the Minkowski-Sum

$$A + B = \{x + y \mid x \in A,\ y \in B\}\,, \qquad A, B \in \mathbb{R}\,.$$

Only the multiplication is some what more laborious. Perhaps it is best to define it first only for *positive* real numbers by

$$A \cdot B = \mathbb{Q}_- \cup \{xy \mid x \in A \cap \mathbb{Q}_+^\times,\ y \in B \cap \mathbb{Q}_+^\times\}\,, \qquad A, B \in \mathbb{R}_+^\times\,,$$

and then extend canonically. The details of this are left to the reader.)

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† **Bernard Placidus Johann Nepomuk Bolzano (1781-1848)** Bernard Placidus Johann Nepomuk Bolzano was born on 5 Oct 1781 in Prague, Bohemia, Austrian Habsburg domain (now Czech Republic) and died on 18 Dec 1848 in Prague, Bohemia (now Czech Republic). Bernard Bolzano was a Czech philosopher, mathematician, and theologian who made significant contributions to both mathematics and the theory of knowledge. Bolzano entered the Philosophy Faculty of the University of Prague in 1796, studying philosophy and mathematics. Bolzano wrote : *My special pleasure in mathematics rested therefore particularly on its purely speculative parts, in other words I prized only that part of mathematics which was at the same time philosophy.*

In the autumn of 1800 he began 3 years of theological study. While pursuing his theological studies he prepared a doctoral thesis on geometry. He received his doctorate in 1804 writing a thesis giving his view of mathematics, and what constitutes a correct mathematical proof. In the preface he wrote: *I could not be satisfied with a completely strict proof if it were not derived from concepts which the thesis to be proved contained, but rather made use of some fortuitous, alien, intermediate concept, which is always an erroneous transition to another kind.*

Two days after receiving his doctorate Bolzano was ordained a Roman Catholic priest. However, as Russ points out that : *He came to realise that teaching and not ministering defined his true vocation.*

Also in 1804, Bolzano was appointed to the chair of philosophy and religion at the University of Prague. Because of his pacifist beliefs and his concern for economic justice, Bolzano was suspended from his position in 1819 after pressure from the Austrian government. Bolzano had not given up without a fight but once he was suspended on a charge of heresy he was put under house arrest and forbidden to publish. Although some of his books had to be published outside Austria because of government censorship, he continued to write and to play an important role in the intellectual life of his country.

Bolzano wrote "Beyträge zu einer begründeteren Darstellung der Mathematik, Erste Lieferung (1810)", the first of an intended series on the foundations of mathematics. Bolzano wrote the second of his series but did not publish it. Instead he decided to : ... *make myself better known to the learned world by publishing some papers which, by their titles, would be more suited to arouse attention.*

Pursuing this strategy he published "Der binomische Lehrsatz ... (1816)" and "Rein analytischer Beweis... (Pure Analytical Proof) (1817)", which contain an attempt to free calculus from the concept of the infinitesimal. He is clear in his intention stating in the preface of the first that the work is : *a sample of a new way of developing analysis.*

Although Bolzano did achieve exactly what he set out to achieve, he did not do this in the short term, his ideas only becoming well known after his death. Russ describes Bolzano's aims in the 1817 paper : *In this work ... Bolzano ... did not wish only to purge the concepts of limit, convergence, and derivative of geometrical components and replace them by purely arithmetical concepts. He was aware of a deeper problem : the need to refine and enrich the concept of number itself.*

The paper gives a proof of the intermediate value theorem with Bolzano's new approach and in the work he defined what is now called a Cauchy sequence. The concept appears in Cauchy's work four years later but it is unlikely that Cauchy had read Bolzano's work. After 1817, Bolzano published no further mathematical works for many years. However, in 1837, he published "Wissenschaftslehre", an attempt at a complete theory of science and knowledge.

Between sometime before 1830 and the 1840s, Bolzano worked on a major work "Grössenlehre". This attempt to put the whole of mathematics on a logical foundation was published in parts, while Bolzano hoped that his students would finish and publish the complete work.

His work on paradoxes "Paradoxien des Unendlichen", a study of paradoxes of the infinite, was published in 1851, three years after his death, by one of his students. The word set appears here for the first time. In this work Bolzano gives examples of 1-1 correspondences between the elements of an infinite set and the elements of a proper subset.

Most of Bolzano's works remained in manuscript and did not become noticed and therefore did not influence the development of the subject. Many of his works were not published until 1862 or later. Bolzano's theories of mathematical infinity anticipated Georg Cantor's theory of infinite sets. It is also remarkable that he gave a function which is nowhere differentiable yet everywhere continuous.