MA221 HOMEWORK ASSIGNMENT 1

Due Date: August 15 (Sun.) by 11:59 pm

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1. **Only the four problems marked with a $*$ will be graded.**
2. Your assignment may be hand-written or typed, but it must be submitted via MS Teams in the form of a **single PDF document**.
3. Unless otherwise stated, you may only use definitions/theorems/facts stated in class.

Notation. $\mathbb{N}_{>0}$ and $\mathbb{Q}_{>0}$ denote the set of positive natural numbers and positive rational numbers, respectively.

Problem A. In (1)-(3), prove the given statements using the ZFC axioms stated in class. Clearly state the axioms (names only, please) as and when you invoke them.

(1$*$) Given objects (or, sets) a and b, the set $(a,b) := \{\{a\},\{a,b\}\}$ has the property that $(a,b) = (c,d)$ if and only if $a = c$ and $b = d$. *Note. In this problem, we have provided a set-theoretic definition of an ordered pair.*

(2) Given sets A and B, there is a set whose elements are precisely all the ordered pairs (a,b), where $a \in A$ and $b \in B$. *Note. This problem defines the Cartesian product, $A \times B$, of A and B.*

(3) Given sets A and B, there is a set whose elements are precisely all the functions from A to B. *Note. Here, we have defined the set that is commonly denoted by B^A. Think why this is consistent with $X \times X$ being denoted by X^2 and $\mathscr{P}(X)$ by 2^X.*

Problem B. Using Peano's axioms, and the properties of $+$, $\times$ and $\leq$ on $\mathbb{N}$ stated in class, prove the following statements.

(1) There is no natural number n such that $0 < n < 1$.

(2$*$) There is no natural number x such that $2x = 1$.

Problem C$*$. Without any reference to real or rational numbers, show that $\mathbb{Z}$ has the least upper bound property.

Problem D. Consider the following relation on $\mathbb{Q}_{>0} \times \mathbb{N}_{>0}$:

$$(a,b) \sim (c,d) \text{ if and only if } a^d = c^b.$$

(a) Show that $\sim$ is an equivalence relation on $\mathbb{Q}_{>0} \times \mathbb{N}_{>0}$.

(b) Recall, from class, that the equivalence relations $(a, b)R(c, d) \iff a + d = c + b$ and $(a, b)R(c, d) \iff ad = cb$ allow us to introduce the operation of subtraction for natural numbers and division for integers, respectively. In the same spirit, what kind of operation can we perform on the equivalence classes of $\sim$ that is unavailable on $\mathbb{Q}_{>0}$?

Problem E∗. Suppose S is an ordered set with the least upper bound property. Let $A, B \subset S$ be sets that are bounded above. Show that if $\sup A < \sup B$, then there is a $b \in B$ such that b is an upper bound of A. Is the conclusion always true if $\sup A \leq \sup B$?

Problem F. In class, we showed that $\mathbb{Q}$ is dense in $\mathbb{R}$, i.e., for every $a, b \in \mathbb{R}$ with $a < b$, there is a rational number r such that $a < r < b$. Is the following set dense in $\mathbb{R}$:

$$\left\{ \frac{p}{q} : p \in \mathbb{Z},\ q \in \mathbb{N},\ \text{and } q \leq 7 \right\}?$$

Provide a complete justification for your answer.