MA221 HOMEWORK ASSIGNMENT 4

Due date: October 19 (Tues.) by 11:59 pm

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1. **Problems for submission.** A(c), B (all parts), C(d).
2. Some of non$*$-ed problems will be discussed during the Wednesday office hours.

Problem A (Problem 24, Chapter 3, Rudin). Let (X, d) be a metric space. Given two Cauchy sequences $\{p_n\}$ and $\{q_n\}$ in X, we say that $\{p_n\} \sim \{q_n\}$ if $\lim_{n\to\infty} d(p_n, q_n) = 0$.

(a) Show that $\sim$ is an equivalence relation on the set of all Cauchy sequences in X.

(b) Let X^* denote the set of all equivalence classes of $\sim$. Given $P, Q \in X^*$, define

$$\Delta(P, Q) = \lim_{n\to\infty} d(p_n, q_n)$$

for some $\{p_n\} \in P$ and $\{q_n\} \in Q$. Show that Δ is well-defined, i.e., for any given choice of $\{p_n\}$, $\{q_n\}$, the limit exists, and the limit is independent of the choice of $\{p_n\} \in P$ and $\{q_n\} \in Q$. Convince yourself that Δ is a metric on X^*.

$(c)*$ Use **one** of the following approaches to show that (X^*, Δ) is a complete metric.
Approach 1. Let $\{P_k\}$ be a Δ-Cauchy sequence of equivalence classes in X^*. Let $\{p_k^n\} \in P_k$. Show that there is a sequence $n_k \in \mathbb{N}$ such that $\{p_k^{n_k}\}$ is a d-Cauchy sequence in X. Show that the equivalence class L of $\{p_k^{n_k}\}$ is the limit of $\{P_k\}$.

Hint. It may help to fist think of the case where each $\{p_k^n\}$ admits a limit l_k in X. What is a good candidate for L in this case?

Approach 2. (i) Consider the map $\Theta : X \to X^*$ given by $p \mapsto \{p_n = p\}$. Show that Θ is an isometry, and $\Theta(X)$ is dense in X.

(ii) Given a metric space Y, show that if every Cauchy sequence in a dense subset $A \subset Y$ converges to a limit in Y, then Y is complete.

(iii) Show that every Cauchy sequence in $\Theta(X)$ converges to a limit in X^*.

Problem B$*$. In class, we mentioned that the discontinuity set D_f of any function $f : \mathbb{R} \to \mathbb{R}$ is an F_σ set. Complete the following steps to produce a proof of this fact.

(a) Given $f : \mathbb{R} \to \mathbb{R}$ and $\alpha > 0$, f is said to be α-*continuous* at $x \in \mathbb{R}$ if there exists a $\delta > 0$ such that for all $y, z \in B(x; \delta)$, $|f(y) - f(z)| < \alpha$. Show that the set $D^\alpha = \{x \in \mathbb{R} : f \text{ is not } \alpha\text{-continuous at } x\}$ is closed, for each $\alpha > 0$.

(b) Show that $D^\alpha \subset D_f$ for any $\alpha > 0$.

(c) Show that

$$D_f = \bigcup_{n=1}^{\infty} D^{\frac{1}{n}}.$$

Problem C. In this problem, we will establish the following result.

Suppose (X, d_X) and (Y, d_Y) are metric spaces, (Y, d_Y) is complete, $E \subset X$ is a dense subset, and $f : E \to Y$ is a uniformly continuous function. Then, there exists a unique continuous function $F : X \to Y$ such that $F|_E = f$.

(a) Establish the uniqueness claim.

(b) Show that uniformly continuous functions map Cauchy sequences to Cauchy sequences. Using this fact, propose a construction for a well-defined function $F : X \to Y$ such that $F|_E = f$.

(c) Show that F is continuous at each $e \in E$. For this, you must show that if $\{x_n\} \subset X \setminus \{e\}$ such that $\lim_{n\to\infty} x_n = e$, then $\lim_{n\to\infty} F(x_n) = f(e)$.

(d)∗ Show that F is continuous at each $x \in X \setminus E$. For this, you must show that if $\{x_n\} \subset X \setminus \{e\}$ such that $\lim_{n\to\infty} x_n = e$, then $\lim_{n\to\infty} F(x_n) = F(e)$. *Hint. This argument is similar to that for* (c)*, however, here you will crucially use the uniform continuity of f on E.*

Problem D. Let (X, d) be a metric space and $A \subset X$ be a nonempty subset. Define $f_A : X \to \mathbb{R}$ as

$$f_A(x) = \inf\{d(x, y) : y \in A\}.$$

(a) Show that f_A is uniformly continuous on X.

(b) There is a closed set $K \subset \mathbb{R}$ such that $\overline{A} = f_A^{-1}(K)$ for any choice of X and A. Determine what K is (and justify your answer).

Problem E. (a) Let $n \in \mathbb{N}$. Produce a function $f : \mathbb{R} \to \mathbb{R}$ that is discontinuous at exactly n points. *Hint. What can you say about $g(x) = xf(x)$, where f is the Dirichlet function discussed in class.*

(b) Consider the function

$$f(x) = \begin{cases} \frac{1}{q}, & x = \frac{p}{q}, \text{ with } p \in \mathbb{Z} \text{ and } q \in \mathbb{N}_{>0} \text{ coprime}, \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Here, we have used the fact that every rational number $x \in \mathbb{Q}$ admits a unique representation of the form p/q, with p and q as described above. Show that f is discontinuous at every rational number, and continuous elsewhere.