MA221 HOMEWORK ASSIGNMENT 5

Due date: November 3 (Wed.) by 11:59 pm

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1. **Problems for submission. B, C (Parts (b)-(d)), D.**
2. Some of non$*$-ed problems will be discussed during the Wednesday office hours.

Problem A. Let $-\infty < a < b < \infty$. Suppose $f : (a, b) \to \mathbb{R}$ is an unbounded differentiable function. Show that $f' : (a, b) \to \mathbb{R}$ is unbounded.

Problem B$*$. In the last assignment, you showed that the set of discontinuities, D_f, of a function $f : \mathbb{R} \to \mathbb{R}$ is an F_σ set, i.e., D_f can be expressed as a countable union of closed sets. In this exercise, you will deduce a similar fact about the set of points of nondifferentiability of a continuous function $f : \mathbb{R} \to \mathbb{R}$. For this, set $F : \mathbb{R} \times (\mathbb{R} \setminus \{0\}) \to \mathbb{R}$ as the function

$$F(x, h) = \frac{f(x + h) - f(x)}{h}.$$

Note that F is continuous on its domain, and differentiability of f at x is equivalent to the existence of $\lim_{h\to 0} F(x, h)$.

Given $\alpha > 0$, we say that f is *α-differentiable* at x if there exist $\beta > 0$, $L \in \mathbb{R}$ such that

$$|F(x, h) - L| \le \alpha \qquad \forall h \text{ satisfying } 0 < |h| < \beta. \tag{1}$$

Let N^α denote the set of points where f is not α-differentiable.

(a) Let $N_f = \{x \in \mathbb{R} : f \text{ is not differentiable at } x\}$. Show that $N_f = \cup_{n\in\mathbb{N}} N^{\frac{1}{n}}$.

Hint. Show that if f is a α-differentiable at x for every $\alpha > 0$, then f is differentiable at x. Here, L is **not** *playing the role of the derivative!*

(b) ~~Express each $N^{1/n}$ as a G_δ set, i.e., as a countable intersection of open sets.~~

Corrected version. Express $\cup_{n\in\mathbb{N}} N^{1/n}$ as a $G_{\delta\sigma}$ set, i.e., as a countable union of a countable intersection of open sets.

Hint. The negation of (1) reads, "For all $\beta > 0$ and $L \in \mathbb{R}$...", which can be written as a double intersection.

Problem C. Complete the following steps to prove the **Inverse Function Theorem** for real-valued functions of one real variable.

(a) Let $f : (a, b) \to \mathbb{R}$ be a strictly monotone function. Show that its inverse is a continuous function on $f((a, b))$.

$(b)*$ Let $f:(a,b)\to\mathbb{R}$ be a strictly monotone function that is differentiable at $x_0\in(a,b)$ with $f'(x_0)\neq 0$. Show that f^{-1} is differentiable at $y_0=f(x_0)$ and

$$(f^{-1})'(y_0)=\frac{1}{f'(x_0)}.$$

$(c)*$ Let $f:(a,b)\to\mathbb{R}$ be a continuously differentiable function, and $f'(x_0)\neq 0$ for some $x_0\in(a,b)$. Then, there exists an open interval $I=(x_0-\delta,x_0+\delta)\subset(a,b)$ such that $f|_I$ is one-to-one with a continuously differentiable inverse g on $J=f(I)$, and

$$g'(y)=\frac{1}{f'(g(y))}\quad\forall y\in J.$$

$(d)*$ Show that the assumption of continuous differentiability is essential in Part (c), i.e., produce a differentiable function f on $\mathbb{R}$ such that $f'(0)\neq 0$, but f is not invertible in any neighborhood of 0.

Problem D$*$. Recall the following function from Assignment 4:

$$f(x)=\begin{cases}\frac{1}{q}, & x=\frac{p}{q},\text{ with } p\in\mathbb{Z}\text{ and } q\in\mathbb{N}_{>0}\text{ coprime},\\ 0, & \text{if } x\in\mathbb{R}\setminus\mathbb{Q}.\end{cases}$$

Using the ε-characterization of Riemann integrability (Theorem A.1 in Lecture 21), show that f is Riemann integrable on $[0,1]$.

Hint. Given $\varepsilon>0$, let $n\in\mathbb{N}$ such that $\frac{1}{n+1}<\varepsilon$. Show that $f(x)<\frac{1}{n+1}$ for all $x\in[0,1]\setminus S_n$, where $S_n=\{\frac{p}{q}:p,q\in\mathbb{N}$ and $1\leq p\leq q\leq n\}$. On the other hand, for $x\in S_n$, $f(x)\leq 1$. Now choose an appropriate partition P of $[0,1]$ so that $U(P,f)<\varepsilon$. Why does this prove the claim?

Problem E. Problem 10, Parts (a)-(c), from Chapter 6 in Rudin's book.

Problem F. (Improper integrals) Suppose $f:(a,b]\to\infty$ and f is Riemann integrable on $[c,b]$ for every $a<c<b$. Define

$$\int_a^b f(x)dx:=\lim_{c\to a^+}\int_c^b f(x)dx$$

if the limit exists. Show that if f is Riemann integrable on $[a,b]$, then this coincides with the integral $\int_a^b f(x)dx$ defined using Darboux sums.