LECTURE - 2 ALMOST COMPLEX STRUCTURES, METRICS, ETC

Now that we know what complex manifolds are, an interesting question is "Given a 2n-real dimensional smooth manifold, is it secretly a complex manifold ?" A much easier question is : "What piece of information on a real vector space allows it to become a complex vector space ?" The answer to this easier question is obtained by knowing how $\sqrt{-1}$ acts on the real vector space V. The properties it should satisfy are :

(1) The action should be $\mathbb{R}$-linear, and
(2) its square is $-Id$.

We define a almost complex structure J on a real vector space V as a real linear map $J : V \to V$ such that $J^2 = -I$. In particular, on $\mathbb{R}^{2n}$, there is a natural almost complex structure given by $J_{std} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$.

Exercise : Prove that such a V is even (real) dimensional and that there exists a basis so that $J = J_{std} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$.

An almost complex manifold is a smooth manifold equipped with a smoothly varying (means that for any smooth vector field X, JX is a smooth vector field, or alternatively, $J = J^i_{\ j} dx^j \otimes \frac{\partial}{\partial x^i}$ in local coordinates where $J^i_{\ j}$ are smooth functions) almost complex structure $J : TM \to TM$. Not every manifold can be given an almost complex structure (even if it is even dimensional). A complex manifold has a natural almost complex structure given by $J(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial y^i}$ and $J(\frac{\partial}{\partial y^i}) = -\frac{\partial}{\partial x^i}$ where $z^i = x^i + \sqrt{-1} y^i$. Why is this well-defined ? If we choose a different set of holomorphic coordinates $\tilde{z}$, then

$$J(\frac{\partial}{\partial \tilde{x}^i}) = J(\frac{\partial}{\partial x^j}\frac{\partial x^j}{\partial \tilde{x}^i} + \frac{\partial}{\partial y^j}\frac{\partial y^j}{\partial \tilde{x}^i}) \tag{0.1}$$

$$= \frac{\partial x^j}{\partial \tilde{x}^i} J(\frac{\partial}{\partial x^j}) + \frac{\partial y^j}{\partial \tilde{x}^i} J(\frac{\partial}{\partial y^j}) \tag{0.2}$$

$$= \frac{\partial x^j}{\partial \tilde{x}^i}\frac{\partial}{\partial y^j} - \frac{\partial y^j}{\partial \tilde{x}^i}\frac{\partial}{\partial x^j}. \tag{0.3}$$

At this point, note that $\frac{\partial z^i}{\partial \bar{\tilde{z}}^j} = 0$ (the Cauchy-Riemann equations). Substituting these in 0.3 we see that

$$J(\frac{\partial}{\partial \tilde{x}^i}) = \frac{\partial}{\partial \tilde{y}^i}$$

and likewise. Therefore, J is a well-defined endomorphism of TM. Please note that NOT every almost complex manifold arises out of a complex manifold. (It is not easy to give such counterexamples though.)

Note that we can consider the complexification of the tangent bundle $T_{\mathbb{C}}M = TM \otimes \mathbb{C}$ (simply replace each tangent space by its complexification). The a.c.s extends in $\mathbb{C}$-linear manner to $T_{\mathbb{C}}M$. Now note that $J\frac{\partial}{\partial z^i} = \sqrt{-1}\frac{\partial}{\partial z^i}$ and $J\frac{\partial}{\partial \bar{z}^i} = -\sqrt{-1}\frac{\partial}{\partial \bar{z}^i}$ where we recall that $\frac{\partial}{\partial z^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} - \sqrt{-1}\frac{\partial}{\partial y^i}\right)$. In fact, one can prove that on any a.c. vector space (V, J), $J_{\mathbb{C}}$ has only two eigenspaces $V^{1,0}, V^{0,1}$ with eigenvalues $\pm\sqrt{-1}$.

The map $L : (T_pM, J) \to (T_p^{1,0}M, \sqrt{-1})$ given by $\frac{\partial}{\partial x^i} \to \frac{\partial}{\partial z^i}$ and $\frac{\partial}{\partial y^i} \to \sqrt{-1}\frac{\partial}{\partial z^i}$ is a complex linear isomorphism between the bundles. (It is well-defined because it can be written in a basis independent manner as $v \to \frac{v-\sqrt{-1}Jv}{2}$.) Note that we have a natural almost complex structure on T^*M as well : $J^*dx^i = -dy^i, J^*dy^i = dx^i$ (why is it well-defined ?). Likewise, $dz^i = dx^i + \sqrt{-1}dy^i$ is an element of $(T^{1,0}M)^*$ (and likewise for $d\bar{z}$). There is a natural complex linear isomorphism $dx^i \to dz^i$, $dy^i \to -\sqrt{-1}dz^i$ between T^*M and $T^{1,0}M$.

Consider the standard manifold $\mathbb{C}$ and $\mathbb{R}^2$. Both have two other natural structures - standard metrics. How do they play with the almost complex structures ? The inner product g (which is also going to be called a metric from now onwards) on $\mathbb{R}^2$ is very special in that J preserves it (after all, J is rotation anticlockwise by 90 degrees), i.e., $g(Jv, Jw) = g(v, w)$. Now there is a Hermitian metric on $T^{1,0}\mathbb{C}$ given by $h(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}) = 1$, i.e., $h = dz \otimes d\bar{z}$ (just like $g = dx \otimes dx + dy \otimes dy$, h is a bilinear map from $T^{1,0} \times T^{0,1} \to \mathbb{C}$ and hence factors uniquely as a linear map from the tensor product and is hence an element of the dual of the tensor product). Writing $dz = dx + \sqrt{-1}dy$, we see that $h = g - \sqrt{-1}(dx \otimes dy - dy \otimes dx) = g - \sqrt{-1}dx \wedge dy$. The real part of h is g and $\sqrt{-1}Im(h)$ seems to be a 2-form $\omega = \frac{\sqrt{-1}}{2}dz \wedge d\bar{z}$. Note that $\omega(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = g(J\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$.

More generally, if M is a complex manifold with a Hermitian metric h on $T^{1,0}M$ (it is a smoothly varying Hermitian inner product), then locally, $h = h_{i\bar{j}}dz^i \otimes d\bar{z}^j$ where $h_{i\bar{j}}$ is a smooth Hermitian positive definite matrix. Its real part is $g = Re(h_{i\bar{j}})(dx^i \otimes dx^j + dy^i \otimes dy^j) + Im(h_{i\bar{j}})(dx^i \otimes dy^j - dy^i \otimes dx^j)$. It is clear that g is symmetric. It defines a well-defined compatible Riemannian metric on M because it can be written in a basis independent manner as $g(v, w) = Re(h(Lv, \bar{Lw}))$ (and this also proves that it is positive-definite). The imaginary part is $-\omega = -Re(h_{i\bar{j}})(dx^i \otimes dy^j - dy^i \otimes dx^j) + Im(h_{i\bar{j}})(dx^i \otimes dx^j + dy^i \otimes dy^j) = \sqrt{-1}\frac{h_{i\bar{j}}}{2}dz^i \wedge d\bar{z}^j$ is a globally-defined real 2-form because $\omega(v, w) = -Imh(Lv, \bar{Lw})$. In fact, as one can see from the local expression, it is a $(1, 1)$-form. Conversely, given a compatible Riemannian metric g, we can define $\omega(X, Y) = g(JX, Y)$ and a Hermitian metric h on $T^{1,0}$ by $h(Lv, Lw) = g(v, w) - \sqrt{-1}\omega(v, w)$.

Now we notice an important fact : A complex manifold is always orientable. To prove this, we simply need to prove that the (real) transition functions have positive Jacobian. The derivative linear map Df_p at a point p is (in the $x, y, \tilde{x}, \tilde{y}$ bases)

$$[Df_p] = \begin{pmatrix} \frac{\partial \tilde{x}^i}{\partial x^j} & \frac{\partial \tilde{x}^i}{\partial y^j} \\ \frac{\partial \tilde{y}^i}{\partial x^j} & \frac{\partial \tilde{y}^i}{\partial y^j} \end{pmatrix}(p) \tag{0.4}$$

Since we can complex linearly extend $Df_p : T_{\mathbb{C},pM} \to T_{\mathbb{C},p}M$ we can choose to express it in the basis $\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^i}$ as

$$[Df]_p = \begin{pmatrix} \frac{\partial \tilde{z}^i}{\partial z^j} & 0 \\ 0 & \frac{\partial \bar{\tilde{z}}^i}{\partial \bar{z}^j} \end{pmatrix}(p) \tag{0.5}$$

whose determinant is positive. In fact, this can be generalised to say that any biholomorphic map between complex manifolds is orientation preserving.

Since compact complex manifolds are orientable in a standard way, we can integrate top forms on them. In particular, given a Riemannian metric g, there is a volume form $\sqrt{\det(g)}dx^1 \wedge dy^1 \wedge dx^2 \wedge dy^2 \ldots$. We can express this form for a compatible Riemannian metric as $\frac{\omega^n}{n!}$. Indeed, it is enough to prove that these two forms coincide at every point in some coordinate system (that can vary from point to point).

Exercise : Show that there is a choice of holomorphic coordinates around a point p so that $h(p) = \sum_i dz^i \wedge d\bar{z}^i$ *(and hence* $g(p) = \sum_i dx^i \otimes dx^i + dy^i \otimes dy^i$ *and* $\omega(p) = \sum_i dx^i \wedge dy^i$*.*

Using the exercise, one can see that the forms are equal. Given a complex submanifold $S \subset M$, and a Hermitian metric h on M, we have an induced Hermitian metric on $T^{1,0}S$ and hence an induced compatible Riemannian metric, and an induced 2-form $\omega_S = i_S^* \omega$. The volume of S equals $\int_S \frac{\omega^s}{s!}$. This is in stark contrast to smooth submanifolds where the volume/area need not be the integral of a globally defined form.

To give more examples, especially on compact manifolds, let us look at a simple comapct manifold : Complex tori. Let $\Lambda \subset \mathbb{C}^n = \mathbb{R}^{2n}$ be a complete lattice, i.e., there is a basis $e_1, e_2, \ldots, e_{2n}$ of $\mathbb{R}^{2n}$ such that every element of Λ is of the form $\sum_i n_i e_i$ where $n_i \in \mathbb{Z}$. Define an equivalence relation $z \sim w$ iff $z - w \in \Lambda$.

Exercise : $\frac{\mathbb{C}^n}{\Lambda}$ *is a compact complex manifold diffeomorphic to* $S^1 \times S^1 \times \ldots$.

One coordinate chart on this complex torus is $z^1, \ldots, z^n$ (just coordinates on $\mathbb{C}^n$). The other charts are obtained from this by simply translating by appropriate elements of Λ. Hence $dz^1, dz^2 \ldots$ are globally well-defined holomorphic 1-forms. Likewise, $\frac{\partial}{\partial z^i}$ are globally well-defined holomorphic vector fields (that are everywhere linearly independent). Hence, define a smooth Hermitian metric by $h(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j} = \delta_{i\bar{j}}$, i.e., $h = \sum_i dz^i \otimes d\bar{z}^i$. It is basically the metric induced from the usual one on $\mathbb{C}^n$.