LECTURES 3 AND 4 (KÄHLER METRICS, COHOMOLOGY, AND EXAMPLES)

1. Lecture 3 (Kähler metrics)

There is a natural Hermitian metric on $T^{1,0}\mathbb{CP}^n$ called the Fubini-Study metric. Here is the unmotivated definition of the same : On U_0, define $h_{i\bar{j}} = \sum_i \frac{dz^i \otimes d\bar{z}^i}{1+|z|^2} - \sum_{i,j} \frac{z^j \bar{z}^i dz^i \otimes d\bar{z}^j}{(1+|z|^2)^2}$. It is not immediately obvious that this is even positive-definite. (Do this as an exercise.) On U_1 let $w^1 = \frac{X^0}{X^1}, w^i = \frac{X^i}{X^1}$ and define $\tilde{h}_{i\bar{j}} = \sum_i \frac{dw^i \otimes d\bar{w}^i}{1+|w|^2} - \sum_{i,j} \frac{w^j \bar{w}^i dw^i \otimes d\bar{w}^j}{(1+|w|^2)^2}$ and likewise for the other coordinate charts. This metric is well-defined. Indeed, when does a collection of coordinate expressions define a metric ? Note that $d\tilde{z}^i = \frac{\partial \tilde{z}^i}{\partial z^j} dz^j$ and hence

$$h = h_{i\bar{j}} dz^i \otimes d\bar{z}^j = \tilde{h}_{k\bar{l}} d\tilde{z}^k \otimes d\bar{\tilde{z}}^l$$

$$= \tilde{h}_{k\bar{l}} \frac{\partial \tilde{z}^k}{\partial z^i} \frac{\partial \bar{\tilde{z}}^l}{\partial \bar{z}^j} dz^i \otimes d\bar{z}^j \tag{1.1}$$

$$\Rightarrow h_{i\bar{j}} = \tilde{h}_{k\bar{l}} \frac{\partial \tilde{z}^k}{\partial z^i} \frac{\partial \bar{\tilde{z}}^l}{\partial \bar{z}^j}. \tag{1.2}$$

Indeed, a calculation shows (do it!) that indeed the above expressions satisfy this change-of-variable condition and therefore define a well-defined Hermitian metric. This metric plays the same role in complex differential geometry as the round metric on the sphere plays in usual Riemannian geometry.

In Riemannian geometry, a nice coordinate system around each point p called normal coordinates can be obtained. In such a coordinate system $g_{ij}(p) = \delta_{ij}$ and $\frac{\partial g_{ij}}{\partial x^k}(p) = 0$. (Therefore, all the Christoffel symbols vanish in this coordinate system at p. However, in general one cannot do better. The second derivatives of g are related to the curvature of g.)

Given a compatible Riemannian metric g on a complex manifold M, a natural question is whether there exists a holomorphic normal coordinate system. If there is a such a system, then clearly $\omega(X,Y) = g(JX,Y)$ also satisfies the same property (because J is standard in holomorphic coordinates). Therefore, $d\omega(p) = 0$, i.e., $\partial\omega(p) = 0$ and $\bar{\partial}\omega(p) = 0$. Since this equation is independent of coordinates and holds for all p, a necessary condition is that $d\omega = 0$. For instance, $h_{i\bar{j}} = (1+|z^2|^2) dz^1 \wedge d\bar{z}^1 + dz^2 \wedge d\bar{z}^2$ does not define a metric in $\mathbb{C}^2$ where such holomorphic normal coordinates can be found. (So this is not a trivial condition.) This condition is actually sufficient :
Using a linear change of coordinates (how ?) we can make sure that z is a coordinate system where $h(p) = \delta_{i\bar{j}} dz^i \otimes d\bar{z}^j$ and

$$b_{i\bar{j}k} = \frac{\partial h_{i\bar{j}}}{\partial z^k}(p) = \frac{\partial h_{k\bar{j}}}{\partial z^i}(p) = b_{k\bar{j}i}.$$

Let $z^i = w^i - \sum_{a,b} \frac{c_{a\bar{i}b}}{2} w^a w^b$ for sufficiently small $|w|$ be a new coordinate system around p (where p corresponds to $z = w = 0$). Then $\frac{\partial z^i}{\partial w^j} = \delta^i_j - \frac{1}{2}\sum_{a,b}(c_{j\bar{i}b} w^b + c_{a\bar{i}j} w^a) = \delta^i_j - \sum_a c_{j\bar{i}a} w^a$. The metric in the w

coordinates near p is

$$\tilde{h}_{ij} = \tilde{h}_{uv}\frac{\partial z^u}{\partial w^i}\frac{\partial \bar{z}^v}{\partial \bar{w}^j} = \tilde{h}_{uv}(\delta^u_i - \sum_a c_{i\bar{u}a}w^a)(\delta^v_j - \sum_a \bar{c}_{j\bar{v}a}w^a) \tag{1.3}$$

Choosing $c = b$ we are done.

A Hermitian metric h on a complex manifold is called Kähler if $d\omega = 0$ (which is equivalent to $\partial\omega = \bar{\partial}\omega = 0$). Here are some examples :

(1) The Euclidean metric on an open subset of $\mathbb{C}^n$.
(2) The standard flat metric on a complex torus.
(3) The Fubini-Study metric on $\mathbb{CP}^n$. (Exercise.)
(4) The metric $h_{i\bar{j}} = \frac{\partial^2}{\partial z^i \bar{z}^j}\ln(1-|z|^2)$ on the open unit disc. This can be done as an exercise but it is easier to note that in this case $\omega = \frac{\sqrt{-1}}{2}\partial\bar{\partial}\ln(1-|z|^2)$ and hence $\partial\omega = 0 = \bar{\partial}\omega$. (This metric plays the same role in complex differential geometry as the hyperbolic metric in Riemannian geometry.)

In each of the above examples the form ω is locally of the form $\sqrt{-1}\partial\bar{\partial}f$ for some real-valued locally defined function f. In fact, this phenomenon is not a coincidence :

Theorem 1.1. *(Local $\partial\bar{\partial}$ lemma) Let h be a smooth Kähler metric on a disc $\mathbb{D}_r \subset \mathbb{C}^n$ of radius r centred at 0. There exists a smooth function $f : \mathbb{D}_r \to \mathbb{R}$ such that $\omega = \sqrt{-1}\partial\bar{\partial}f$.*

Before we prove this statement, let us look at another point. How can one come up with examples of two forms ω (let alone $(1,1)$-forms of the type $\omega = \frac{\sqrt{-1}}{2}h_{i\bar{j}}dz^i \wedge d\bar{z}^j$ where h is a positived-definite matrix) that are closed, i.e., $d\omega = 0$? One way is to take an exact form $\omega = d\eta$ (and hence $d\omega = d^2\eta = 0$). This raises a couple of questions ?

(1) Is every closed k-form exact ? No. Indeed, take $\eta = \frac{-ydx+xdy}{x^2+y^2}$ There is no smooth function f on $\mathbb{R}^2 - \{0\}$ such that $\eta = df$. Indeed, if there was one, then $\int_{x^2+y^2=1}\eta = \int_{S^1} df = 0$ (by Stokes) but $\int_{S^1}\eta = 2\pi$ (as can be calculated easily).
(2) At least, is a Kähler form ω exact ? Certainly never on compact complex manifolds. Indeed, if $\omega = d\eta$, then $\frac{\omega^n}{n!} = \frac{d\eta\omega^{n-1}}{n!} = \frac{d(\eta\omega^{n-1})}{n!}$ whose integral is 0 by Stokes. But, the integral of the left hand side is the volume of the manifold which is always positive.

The above being said, here are a few related points.

(1) (Poincaré lemma) : A closed k-form (k¿0) on an open ball in $\mathbb{R}^n$ (or more generally, on a convex subset of $\mathbb{R}^n$) is exact. While this lemma is not terribly hard to prove, we shall only prove it for 1-forms $\eta = \eta_i dx^i$ such that $d\eta = \sum_{i<j}(\frac{\partial\eta_i}{\partial x^j} - \frac{\partial\eta_j}{\partial x^i})dx^i \wedge dx^j = 0$. Indeed, define $f(x) = \int_0^1 \eta_i(tx)x^i dt$. Now

$$\frac{\partial f}{\partial x^j} = \int_0^1 \frac{\partial\eta_i}{\partial x^j}(tx)tx^i + \eta_j(tx)dt$$
$$= \int_0^1 (\frac{\partial\eta_j}{\partial x^i}(tx)tx^i + \eta_j(tx))dt = \eta_j(x). \tag{1.4}$$

2. Lecture 4 (Kähler metrics)

(1) In general, we can try to measure how much forms fail from being exact (and that information will help us detect the number of "holes" in our manifold like in the example above) using a tool called De Rham cohomology. Indeed, let $C^k(M)$ be the (infinite-dimensional) space of smooth closed k-forms and $dC^{k-1}(M)$ the subspace of smooth exact k-forms. We can take a quotient of these spaces, i.e., $H^k(M) = \frac{C^k(M)}{dC^{k-1}(M)}$ to get a vector space consisting of equivalence classes $[\omega]$ (basically closed forms upto exactness) called the k^{th} De Rham cohomology of M. It turns out that these vector spaces (or groups as they are called sometimes) are finite-dimensional if M is a compact manifold.
Poincaré 's lemma can be stated as $H^k(\mathbb{B} \subset \mathbb{R}^n) = 0 \ \forall \ k > 0$. For $k = 0$, $df = 0$ implies that f is a constant on each component. Hence, $dim(H^0(M))$ is simply the number of connected components of M. (So to prove something is connected, one can try to calculate the 0^{th} De Rham group and prove it is one-dimensional.) Clearly, $H^k(M) = 0$ when $k > dim(M)$. For a compact Kähler manifold, $H^2(M) \neq \{0\}$ because $[\omega] \neq [0]$ where ω is the Kähler form. (In fact, a standard method calculating De Rham cohomology shows that the Hopf surface has $H^2 = 0$ and hence cannot be Kähler.) Here is an exercise :
Exercise : Calculate the De Rham cohomology of a torus $S^1 \times S^1$ ... from first principles.

(2) A compact oriented manifold has $dimH^{dim(M)}(M) = 1$. Also, a non-trivial result (Poincarè duality) shows that $dim(H^k(M)) = dim(H^{dim(M)-k}(M))$ when M is a compact orientable manifold. In general, there are many techniques to compute the De Rham cohomology. So it is a useful tool to distinguish between manifolds.

(3) On a complex manifold, there are other cohomology groups : We can ask whether $\bar{\partial}$-closed forms are exact or not for instance. So we have the Dolbeault cohomology groups $H^{p,q}_{\bar{\partial}}(M) = \frac{\bar{\partial} \text{ closed } (p,q) \text{ forms}}{\bar{\partial} \text{ exact } (p,q) \text{ forms}}$. For Kähler manifolds, it turns out that $H^k \equiv \oplus_{p+q=k} H^{p,q}$, and that $H^{p,q} = H^{q,p}, H^{n-p,n-q} = H^{p,q}$. So in particular, for Kähler manifolds, $b_1 = dim(H^1) = dim(H^{1,0}) + dim(H^{0,1}) = 2dimH^{1,0}$ is even. Also, $dimH^{1,1} > 0$ for a Kähler manifold because the Kähler form is in it.

(4) There is a $\bar{\partial}$ Poincaré lemma for a ball in $\mathbb{C}^n$. It is actually quite non-trivial to prove even for $(0,1)$-forms on $\mathbb{C}$. Indeed, if $\eta = gd\bar{z}$ on a disc, then it turns out that $f = \frac{1}{2\pi\sqrt{-1}} \int \frac{g(w)}{w-z} dw \wedge d\bar{w}$ satisfies $\bar{\partial} f = \eta$ on a slightly smaller disc. This itself requires a trick (partition-of-unity) to prove. The proof is in Griffiths and Harris.

(5) Hence, if $d\omega = 0$ where ω is a $(1,1)$-form on a ball in $\mathbb{C}^n$, then $\omega = d\eta$ where $\eta = \eta^{1,0} + \eta^{0,1}$. Hence, $\partial\eta^{1,0} = 0, \bar{\partial}\eta^{0,1} = 0$. Thus, $\omega = \partial\eta^{0,1} + \bar{\partial}\eta^{1,0} = \partial\bar{\partial} f_1 + \bar{\partial}\partial f_2$. Now, $\eta^{1,0} + \eta^{0,1} = \eta$ is a real form and hence $\eta^{1,0} = \eta^{\bar{0},1}$. Hence, f_2 can be chosen to be equal to $\bar{f_1}$. Thus, $\omega = \sqrt{-1}\partial\bar{\partial} f$ where f is a real-valued function. This proves the local $\partial\bar{\partial}$ lemma.

On a compact manifold, is there a version of the $\partial\bar{\partial}$ lemma globally ? Thankfully, yes.

Theorem 2.1. *If ω is a d-closed real $(1,1)$-form, then any other d-closed ω' in $[\omega]$, i.e., $\omega' = \omega + d\eta$ (where η is a real 1-form) can be written as $\omega' = \omega + \sqrt{-1}\partial\bar{\partial}\phi$ where ϕ is a smooth globally defined real valued function.*

The proof of this theorem uses an analytic tool (the Hodge theorem) and we will not do it (at least right now).

We are in a position to state the Calabi volume conjecture. (Calabi made many interrelated conjectures. One of them is this one.) : Given a smooth function f and a Kähler metric ω such that $\int_M e^f \omega^n = \int_M \omega^n$, there exists a unique Kähler metric ω' in the same De Rham cohomology class of ω such that the volume form of ω' is $e^f\omega^n$. In other words, prove that there is upto constants a unique smooth function ϕ satisfying the complex Monge-Ampère equation

$$(\omega + \sqrt{-1}\partial\bar{\partial}\phi)^n = e^f \omega^n. \tag{2.1}$$

Is there a systematic way of coming up with the Fubini-Study metric ? It turns out that there is at least a systematic way of coming up with some closed $(1,1)$-forms (that are not necessarily exact). So at least that gives us some hope. Indeed, let L be a holomorphic line bundle. Recall that it means that $L = \cup_p L_p$ is a complex manifold such that L_p is a complex 1-d vector space at every p, and that locally, there exists a holomorphically varying basis $s_\alpha : U_\alpha \to L$. A smooth Hermitian metric h on L is simply a smoothly varying collection of inner products h_p, i.e., the local functions $h_\alpha(p) = h_p(s_\alpha(p), s_\alpha(p))$ are smoothly varying. Moreover, if we change our local basis to s_β such that $s_\alpha = g_{\beta\alpha}s_\beta$ where $g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathbb{C}^*$ is a holomorphic function, then $h_\alpha = h(g_{\beta\alpha}s_\beta, g_{\beta\alpha}s_\beta) = |g_{\beta\alpha}|^2 h_\beta$. Consider the expression $\Theta_\alpha = \bar{\partial}\partial \ln h_\alpha = \bar{\partial}\partial \ln(h_\beta) = \Theta_\beta$. So using a Hermitian metric on a holomorphic line bundle L, we can produce a globally defined $(1,1)$-form $\Theta = \bar{\partial}\partial \ln h_\alpha$ that is clearly d-closed. This $(1,1)$-form is called the "curvature form" of the "Chern connection" of h. This form is purely imaginary.