NOTES ON LINEAR ODE AND STABILITY

1. Why to care about differential equations

(1) All laws of nature are differential equations (whether ODE or PDE). Even modelling simple things like "There are 25 rabbits and 13 foxes; after observing them for a month, predict how the situation will be like if there were 15 foxes to begin with" boil down to differential equations.
(2) The question of "Can we draw a map of Bangalore on a piece of paper such that is to scale ? That is, 2 cm in the map is maybe 1 km in Bangalore ?" (No) This has to do in a deep manner with certain kinds of partial differential equations. This is the beginning of the study of Differential Geometry (geometry on curved surfaces like that of the earth).
(3) Find positive integers x, y, z so that $\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} = 4$. This is a shockingly difficult problem. The smallest integers satisfying this are 80 digit numbers ! This has deep connections to an ODE (involving complex numbers no less!) whose solutions are called Weierstrass elliptic functions. You can read more about this on `https://www.quora.com/How-do-you-find-the-positive-integer-solutions-to-frac-x-y%2Bz-%2B-frac-y-z%2Bx-%2B-frac-z-x%2By-4/answer/Alon-Amit`

The examples above cannot be understood by simply knowing a collection of techniques to solve simple differential equations. What we need is a unified *theory*. For instance, one question which mathematicians try to answer is "Does a solution to this differential equation exist ? Is it unique ?" - Note that one can write innocent looking differential equations for which you can prove that there are *no* solutions (leave alone actually finding a formula for the solutions using techniques you learn in college).

So our aim in this set of lectures is to *rigorously* study some aspects of the *theory* of differential equations, in particular, linear differential equations.

2. A seemingly silly example

Q : Suppose $D = (0,1) \cup (2,3)$. Find a differentiable function $f : D \to \mathbb{R}$ satisfying $f^{'}(x) = 0 \ \forall \ x \in D$.

Ans : One might be tempted to say that f is a constant. This is not true ! $f = 1$ *on* $(0,1)$ and $f = 2$ *on* $(2,3)$ is a non-constant solution ! So firstly, the solution (or the lack of thereof) of a differential equation might depend on the *domain*. In fact, knowing all solutions of this differential equation tells us that D is not connected ! In other words, one can potentially shed light on the *shape* of a region by studying solutions of differential equations on it !

So the answer is $f = c_1$ on $(0,1)$ and $f = c_2$ on $(2,3)$. How does one prove this ? There are two ways. In each of these, let us prove that f is a constant on $(0,1)$ (the case of $(2,3)$ is similar).

(1) (MVT) : Suppose $a \in (0,1)$. Since f is assumed to be differentiable on $(0,1)$, it is also continuous on it. Hence it is continuous on $[\frac{1}{2}, a]$ (or $[a, \frac{1}{2}]$ if $a < \frac{1}{2}$). Then $f(a) - f(\frac{1}{2}) = f^{'}(\theta)(a - \frac{1}{2})$ for some θ. This is the content of the Mean Value Theorem. Hence $f(a) = f(\frac{1}{2}) \ \forall \ a \in (0,1)$.

(2) (FTC) : If f is continuous on $[a,b]$ and there is a differentiable function F so that $F^{'} = f$ on $[a,b]$, then $F(b) - F(a) = \int_a^b f(x)dx$. Since $f^{'} = 0$ is continuous, $\int_{1/2}^a f^{'}(x)dx = f(a) - f(1/2) = 0$.

3. A slightly less silly one

Many situations in real life (population of humans or rabbits or anything else, radioactivity, chemical reactions, etc) involve the rate of change being proportional to the quantity present (the more rabbits, the faster their number grows). This is modelled by the differential equation $\frac{dy}{dx} = ay$ for some constant real number a.

Q : Find all differentiable functions $y : \mathbb{R} \to \mathbb{R}$ such that $\frac{dy}{dx} = ay$.
This is a question you can all solve but we want to be *rigorous* and extract *concepts* out of the solution. The claim is that all solutions are of the form $y = Ce^{ax}$. Indeed, $\frac{d(ye^{-ax})}{dx} = 0$ and hence (since $\mathbb{R}$ is connected), $y = Ce^{ax}$. More importantly, if $a = 0$, $y = C$, if $a > 0$, y increases very quickly (faster than any polynomial), and if $a < 0$, it decreases quickly. y is never 0 unless it is so to begin with.

So the initial condition $y(0)$ determines whether y is positive always or negative always. When $a < 0$, the solution tends to come back to equilibrium. When $a > 0$, it tends to move away from equilibrium. The solution $y = 0$ is hence a stable ($a < 0$), unstable ($a > 0$) or neutral equilibrium.

4. A far less silly one

Here is another example : In a realistic population model, growth cannot be exponential forver. After some time, the lack of resources will decrease the rate of growth and it stabilises to some "carrying capacity". So a more realistic model (called the Logistic model) is

$$\frac{dy}{dx} = ay(1 - \frac{y}{K}) \tag{4.1}$$

on $[0, \infty)$. One can solve this explicitly using standard techniques

$$y = \frac{K}{1 + (\frac{K}{y_0} - 1)e^{-ax}}$$

This sort of a function is called the Logistic function/sigmoid function (and it comes up in other areas of life such as statistics and machine learning). But this is not the point. We want to study qualitative behaviour. If $y_0 = 0$ or $y_0 = K$, then $y = y_0$ (the constant solution) is *a* solution and hence *the* solution with those initial conditions (the theorem of existence and uniqueness of solutions to ODE). What if $y_0 = K \pm$ *a small amount* ? Suppose $y_0 = K - h$ where $h > 0$ is small. Then the right hand side is negative to begin with. Thus y decreases initially. This means the right hand side becomes slightly less negative and so on. It turns out that in this particular case, the right hand side never actually reaches zero but becomes less and less negative (y keeps decreasing towards K). Likewise for the other cases. Slightly more precisely (but still not completely rigorous), if we are given something like $\frac{dy}{dx} = F(y)$ where $F(y_0) = 0$, then writing $y = y_0 + h$ where h is small, $\frac{dy}{dx} = \frac{dh}{dx}$ is approximately $F^{'}(y_0)h$ (the first order Taylor expansion). Thus, h is approximately $h_0 e^{F^{'}(y_0)x}$. So if $F^{'}(y_0) < 0$, this perturbation seems to die down eventually and the equilibrium point y_0 is stable. Otherwise, it eventually seems to grow and the point seems to be unstable. In this particular case, we can verify this rigorously using the explicit formula.

The bottom line is that qualitative analysis of $\frac{dy}{dx} = F(y)$ seems to rely on equations like $\frac{dy}{dx} = ay$. If $F(y_e) = 0$, then a small perturbation to this equilibrium might die out (stable) or grow (unstable) depending on the sign of $F^{'}(y_0)$. But to make this rigorous is quite challenging. It is also helpful in these cases to draw graphs of y vs x.

5. Systems of ODE

Suppose we consider a model with two unrelated species : Pigeons and rabbits. They obey $\frac{dy_1}{dx} = a_1 y$ and $\frac{dy_2}{dx} = a_2 y$. Clearly, the unique solution to this system is $y_1 = (y_1)_0 e^{a_1 x}$ and $y_2 = (y_2)_0 e^{a_2 x}$. In other words, their populations grow or decay irrespective of the other one.

Here is a more interesting system

$$\frac{dy_1}{dx} = 2y_1 + y_2$$
$$\frac{dy_2}{dx} = y_1 + 2y_2$$

If we can change the variables from y_i to z_i such that the new equation looks like $\frac{dz_1}{dx} = \lambda_1 z_1$ and $\frac{dz_2}{dx} = \lambda_2 z_2$ (they "decouple"), then we will be in great shape. So let's try $z_1 = ay_1 + by_2$ and $z_2 = cy_1 + dy_2$. Then

$$\frac{dz_1}{dx} = a(2y_1 + y_2) + b(y_1 + 2y_2) = (2a + b)y_1 + (a + 2b)y_2$$
$$\frac{dz_2}{dx} = c(2y_1 + y_2) + d(y_1 + 2y_2) = (2c + d)y_1 + (c + 2d)y_2$$

We want the RHS to be proportional to z_1, z_2 respectively. Thus

$$2a + b = \lambda_1 a, a + 2b = \lambda_1 b$$
$$2c + d = \lambda_2 c, c + 2d = \lambda_2 d$$

Therefore, $(\lambda_1 - 2)a = b$, $(\lambda_1 - 2)b = a$. Hence, $(\lambda_1 - 2)^2 = 1$ and likewise for λ_2. Thus $\lambda_1 = 1$ and $\lambda_2 = 3$. Once we get these λ_i, clearly, $a = 1, b = -1$ and $c = 1, d = 1$ do the job.

Do we see something familiar in the above example ? We can write all of these things in the language of matrices. Indeed,

$$\frac{d\vec{y}}{dx} = A\vec{y}$$

where $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. The change of variables can be written as $\vec{z} = P\vec{y}$ where $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Note that $\vec{y} = P^{-1}\vec{z}$. Therefore,

$$P^{-1}\frac{d\vec{z}}{dx} = AP^{-1}\vec{z} \Rightarrow \frac{d\vec{z}}{dx} = PAP^{-1}\vec{z}$$

So if we are lucky enough to find P so that it is invertible and PAP^{-1} is a diagonal matrix consisting of λ_1, λ_2, we will be in great shape. Indeed, if $PAP^{-1} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, then $z_1 = (z_0)_1 e^{\lambda_1 x}$, $z_2 = (z_0)_2 e^{\lambda_2 x}$. Now, $\vec{y} = P^{-1}\vec{z} = P^{-1}\begin{bmatrix} e^{\lambda_1 x} & 0 \\ 0 & e^{\lambda_2 x} \end{bmatrix} P\vec{y}_0$.

This process is called "diagonalisation". The λ_i are called "eigenvalues" of A. The columns of P^{-1} are called "eigenvectors", i.e., an eigenvector of a matrix A corresponding to an eigenvalue λ is a vector v such that $Av = \lambda v$. So if given a 2×2 matrix A, if we manage to find 2 linearly independent eigenvectors v_1, v_2, then we can solve $\frac{d\vec{y}}{dx} = A\vec{y}$ without any problem. Unfortunately, this is not always possible. For example, the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has only one linearly independent eigenvector of the form $t\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for any $t \in \mathbb{R}$. In general, how does one determine the eigenvalues of a matrix A ? Well, an eigenvalue is a (real or complex) number λ so that $Av = \lambda v$. This means that $(A - \lambda I)v = 0$. This means that $A - \lambda I$ is not invertible and hence $\det(A - \lambda I) = 0$. In fact, if $\det(A - \lambda I) = 0$, then there is a vector v so that $(A - \lambda I)v = 0$. Indeed, for 2×2 matrices, $av_1 + bv_2 = \lambda v_1$ and hence assuming $b \neq 0$, choosing $v_2 = \frac{(\lambda - a)}{b}$, $v_1 = 1$, we see that $(A - \lambda I)v = 0$.

Sometimes, the roots of this "characteristic" polynomial $\det(A - \lambda I)$ can be complex numbers. For instance, if $A = \begin{bmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & -1 \end{bmatrix}$, then the eigenvalues are $\pm\sqrt{-1}$. How can we interpret this in terms of differential equations ? It just means that z_2 and z_1 are complex linear combinations of y_1, y_2. Also, $\frac{dy}{dx} = (a + ib)y$ can be solved as $y = y_0 e^{(a+ib)x}$. Indeed, $\frac{d(ye^{-(a+ib)x})}{dx} = 0$. So setting the real and imaginary parts to 0 and using our previous discussion that real-valued functions on $\mathbb{R}$ whose derivative is 0 are constant, we see that $y = y_0 e^{(a+ib)x} = e^{ax}(\cos(bx) + i\sin(bx))$. So alternatively, in the case where the eigenvalues are complex, y_1, y_2 are real linear combinations of cosines and sines.

By the way, the second order equation $\frac{d^2y}{dx^2} = -y$ can be written as a system of two first order equations by introducing a new variable $v = \frac{dy}{dx}$ and writing $\frac{dy}{dx} = v$, $\frac{dv}{dx} = -y$. Thus $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ in this case. The eigenvalues are $\pm i$ and 2 linearly independent eigenvectors are $\begin{bmatrix} 1 \\ i \end{bmatrix}, \begin{bmatrix} 1 \\ -i \end{bmatrix}$. Hence $P^{-1} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$. This means that $P = \begin{bmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{1}{2} & \frac{i}{2} \end{bmatrix}$ Thus, $\vec{y} = P^{-1}\begin{bmatrix} e^{ix} & 0 \\ 0 & e^{-ix} \end{bmatrix} P\vec{y}_0 = \begin{bmatrix} (y_1)_0\cos(x) + (y_2)_0\sin(x) \\ -(y_1)_0\sin(x) + (y_2)_0\cos(x) \end{bmatrix}$

In any case, whether real or complex, we are still faced with the questions : "How can you know if a matrix is diagonalisable ?", "Even if it is, how can you calculate the eigenvalues and eigenvectors ?", "If it is not diagonalisable, how can you solve the differential equation $\frac{d\vec{y}}{dx} = A\vec{y}$?" (We shall restrict our attention to 2×2 real matrices. But this discussion can be generalised to $n \times n$ complex matrices too.)

We first prove this result :

Lemma 5.1. *If A has* distinct *eigenvalues, i.e., $\lambda_1 \neq \lambda_2$, then A is diagonalisable, i.e., there are two linearly independent vectors v_1, v_2 (which may have complex entries) such that $Av_i = \lambda_i v_i$.*

Proof. Indeed, if there are distinct eigenvalues, there are eigenvectors v_1, v_2 corresponding to them. We just need to prove that they are linearly independent. Indeed, if $c_1v_1 + c_2v_2 = 0$, then $c_1Av_1 + c_2Av_2 = 0$ which means that $c_1\lambda_1v_1 + c_2\lambda_2v_2 = 0$. Thus, $\lambda_1c_2 = \lambda_2c_2$. If $c_2 \neq 0$, then we have a contradiction. Otherwise, $\lambda_1c_1 = \lambda_2c_1$. Again, we have the same problem. ◻

Another nice result is :

Lemma 5.2. *If $A = A^T$, i.e., A is a symmetric real 2×2 matrix, then it is diagonalisable.*

Proof. In this case, suppose λ_1 is one eigenvalue with a corresponding eigenvector $v_1 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$. Then $v_2 = \begin{bmatrix} \beta \\ -\alpha \end{bmatrix}$ is perpendicular to v_1 (and hence linearly independent). Now $v_1^T A v_2 = (A^T v_1)^T v_2 = (Av_1)^T v_2 = \lambda_1 v_1^T v_2 = 0$. Thus, Av_2 is perpendicular to v_1. Hence, it must point along v_2. Therefore $Av_2 = \lambda_2 v_2$. This means, the linearly independent vectors v_1, v_2 are eigenvectors. □

All of these results still do not seem to address the case when something is not diagonalisable like $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. In this case, one can still solve the differential equation. Indeed, $\frac{dy_2}{dx} = y_2$ and hence $y_2 = (y_2)_0 e^x$. Now $\frac{dy_1}{dx} = y_1 + y_2 = y_1 + (y_2)_0 e^x$. This we can solve using an integrating factor. Indeed, $y_1 e^{-x} - (y_1)_0 = (y_2)_0 x$. Thus $y_1 = (y_1)_0 e^x + (y_2)_0 x e^x$. So we will be in good shape even if we manage to find a P so that $PAP^{-1} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ (note that if $\lambda_1 \neq \lambda_2$, it is diagonalisable; so this problem can arise only when the eigenvalues are equal). In this case, $\vec{y} = P^{-1} \begin{bmatrix} e^{\lambda x} & xe^{\lambda x} \\ 0 & e^{\lambda x} \end{bmatrix} P \vec{y}_0$. Saying that A can be brought to this form is the same as saying that the columns of P^{-1} consist of an eigenvector v and another vector w such that $Aw = \lambda w + v$. Indeed,

Lemma 5.3. *If $\lambda_1 = \lambda_2 = \lambda$, and there is only one linearly independent eigenvector v of A, then there exists another vector w such that $Aw = \lambda w + v$.*

Proof. Suppose $v = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$. Let $\tilde{v} = \begin{bmatrix} -\beta \\ \alpha \end{bmatrix}$. Then $\tilde{v}$ is perpendicular to v. Hence, we can resolve any vector w into components along v and $\tilde{v}$. (This needs a proof but we shall assume this.) Now $(A - \lambda I)\tilde{v} \neq 0$ because $\tilde{v}$ is independent of v and A is assumed to not be diagonalisable. However, $(A-\lambda I)\tilde{v} = c_1 v + c_2 \tilde{v}$ which means that $(A-\lambda I)^2 \tilde{v} = c_2 (A - \lambda I)\tilde{v}$ which means that $(A-\lambda I)\tilde{v} = \alpha v$. Thus defining $w = \frac{\tilde{v}}{\alpha}$, we see that $(A - \lambda)w = v$. □

In any case, if we take a linear system of ODE, it appears that we can more or less solve it. Moreover, if the eigenvalues of A are < 0, then as $x \to \infty$, $y \to 0$. This is an important observation. Suppose we have a system of nonlinear ODE like a competing species model :

$$\frac{dy_1}{dx} = y_1 - y_1 y_2$$
$$\frac{dy_2}{dx} = y_2 - y_1 y_2$$

then a natural question is "Is there an equilibrium? i.e., an initial condition for rabbits and squirrels, so that as soon as a rabbit is born, one dies because of competition/fighting with squirrels, and likewise for squirrels ? Is this equilibrium stable ? That is, if I introduce an extra rabbit and observe for a long time, will the numbers return to equilibrium or will the rabbits dominate the squirrels ?"

To answer this question, set the RHS to 0. Indeed, we get the equilibria as $y_1 = 0, y_2 = 0$ and $y_1 = y_2 = 1$. Now $y_1 = y_2 = 1$ is neither stable nor unstable because, suppose $y_1 = 1 + h_1$, $y_2 = 1 + h_2$ where h_1, h_2 are small, then the equation is approximately (neglecting higher order terms), $\frac{dh_1}{dx} = -h_2, \frac{dh_2}{dx} = -h_1$. Thus the matrix is $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ whose eigenvalues are ± 1.

More generally, if $\frac{d\vec{y}}{dx} = \vec{F}(\vec{y})$, and if $\vec{y}_e$ is a point of equilibrium, i.e., $\vec{F}(\vec{y}_e) = 0$, then writing

$\vec{y} = \vec{y}_e + \vec{h}$ where $\vec{h}_0$ is small, $\frac{d\vec{h}}{dx}$ is approximately $\frac{\partial \vec{F}}{\partial y_1} h_1 + \frac{\partial \vec{F}}{\partial y_2} h_2$. If the eigenvalues of the derivative matrix are < 0, then the equilibrium is stable. Once again, making this rigorous is quite challenging.