MA 229/MA 235 - Lecture 12

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Recap

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- Raised a question about inverse images and images of smooth maps between manifolds.

Immersions and submersions

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- The idea is to use a partition-of-unity to "patch them together".

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