MA 229/MA 235 - Lecture 19

IISc

Recap

Recap

- Discussed canonical coordinates for vector fields.

Recap

- Discussed canonical coordinates for vector fields.
- Started motivating the Lie bracket.

Lie bracket: Definition

Lie bracket: Definition

- The expression $-\frac{\partial Y^i}{\partial x^k}(p)X^k(p) + \frac{\partial X^i}{\partial x^k}(p)Y^k(p)$ actually defines a vector field (why?)

Lie bracket: Definition

- The expression $-\frac{\partial Y^i}{\partial x^k}(p)X^k(p) + \frac{\partial X^i}{\partial x^k}(p)Y^k(p)$ actually defines a vector field (why?)
- Is there an coordinate-invariant way

Lie bracket: Definition

- The expression $-\frac{\partial Y^i}{\partial x^k}(p)X^k(p) + \frac{\partial X^i}{\partial x^k}(p)Y^k(p)$ actually defines a vector field (why?)
- Is there an coordinate-invariant way of defining this vector field?

Lie bracket: Definition

- The expression $-\frac{\partial Y^i}{\partial x^k}(p)X^k(p) + \frac{\partial X^i}{\partial x^k}(p)Y^k(p)$ actually defines a vector field (why?)
- Is there an coordinate-invariant way of defining this vector field?
- Def:

Lie bracket: Definition

- The expression $-\frac{\partial Y^i}{\partial x^k}(p)X^k(p) + \frac{\partial X^i}{\partial x^k}(p)Y^k(p)$ actually defines a vector field (why?)
- Is there an coordinate-invariant way of defining this vector field?
- Def: Let X, Y be smooth vector fields on a manifold M.

Lie bracket: Definition

- The expression $-\frac{\partial Y^i}{\partial x^k}(p)X^k(p) + \frac{\partial X^i}{\partial x^k}(p)Y^k(p)$ actually defines a vector field (why?)
- Is there an coordinate-invariant way of defining this vector field?
- Def: Let X, Y be smooth vector fields on a manifold M. Then $[X, Y]_p(f) = X_p(Y(f)) - Y_p(X(f))$ is a vector field on M

Lie bracket: Definition

- The expression $-\frac{\partial Y^i}{\partial x^k}(p)X^k(p) + \frac{\partial X^i}{\partial x^k}(p)Y^k(p)$ actually defines a vector field (why?)
- Is there an coordinate-invariant way of defining this vector field?
- Def: Let X, Y be smooth vector fields on a manifold M. Then $[X, Y]_p(f) = X_p(Y(f)) - Y_p(X(f))$ is a vector field on M called the Lie bracket of X and Y.

Lie bracket: Definition

- The expression $-\frac{\partial Y^i}{\partial x^k}(p)X^k(p) + \frac{\partial X^i}{\partial x^k}(p)Y^k(p)$ actually defines a vector field (why?)
- Is there an coordinate-invariant way of defining this vector field?
- Def: Let X, Y be smooth vector fields on a manifold M. Then $[X, Y]_p(f) = X_p(Y(f)) - Y_p(X(f))$ is a vector field on M called the Lie bracket of X and Y.
- Lemma (proof by calculation):

Lie bracket: Definition

- The expression $-\frac{\partial Y^i}{\partial x^k}(p)X^k(p) + \frac{\partial X^i}{\partial x^k}(p)Y^k(p)$ actually defines a vector field (why?)
- Is there an coordinate-invariant way of defining this vector field?
- Def: Let X, Y be smooth vector fields on a manifold M. Then $[X, Y]_p(f) = X_p(Y(f)) - Y_p(X(f))$ is a vector field on M called the Lie bracket of X and Y.
- Lemma (proof by calculation): The Lie bracket genuinely defines a vector field

Lie bracket: Definition

- The expression $-\frac{\partial Y^i}{\partial x^k}(p)X^k(p) + \frac{\partial X^i}{\partial x^k}(p)Y^k(p)$ actually defines a vector field (why?)
- Is there an coordinate-invariant way of defining this vector field?
- Def: Let X, Y be smooth vector fields on a manifold M. Then $[X, Y]_p(f) = X_p(Y(f)) - Y_p(X(f))$ is a vector field on M called the Lie bracket of X and Y.
- Lemma (proof by calculation): The Lie bracket genuinely defines a vector field whose components are given above.

Lie bracket: Properties

Lie bracket: Properties

- $[X, Y] = -[Y, X]$.

Lie bracket: Properties

- $[X, Y] = -[Y, X]$.
- $[X, Y]$ is multi-linear in X, Y.

Lie bracket: Properties

- $[X, Y] = -[Y, X]$.
- $[X, Y]$ is multi-linear in X, Y.
- It is not associative! That is

Lie bracket: Properties

- $[X, Y] = -[Y, X]$.
- $[X, Y]$ is multi-linear in X, Y.
- It is not associative! That is $[X, [Y, Z]] \neq [[X, Y], Z]$!

Lie bracket: Properties

- $[X, Y] = -[Y, X]$.
- $[X, Y]$ is multi-linear in X, Y.
- It is not associative! That is $[X, [Y, Z]] \neq [[X, Y], Z]$! Indeed, $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (Jacobi's identity).

Lie bracket: Properties

- $[X, Y] = -[Y, X]$.
- $[X, Y]$ is multi-linear in X, Y.
- It is not associative! That is $[X, [Y, Z]] \neq [[X, Y], Z]$! Indeed, $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (Jacobi's identity).
- Any vector space equipped with

Lie bracket: Properties

- $[X, Y] = -[Y, X]$.
- $[X, Y]$ is multi-linear in X, Y.
- It is not associative! That is $[X, [Y, Z]] \neq [[X, Y], Z]$! Indeed, $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (Jacobi's identity).
- Any vector space equipped with such a "bracket"

Lie bracket: Properties

- $[X, Y] = -[Y, X]$.
- $[X, Y]$ is multi-linear in X, Y.
- It is not associative! That is $[X, [Y, Z]] \neq [[X, Y], Z]$! Indeed, $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (Jacobi's identity).
- Any vector space equipped with such a "bracket" is called a Lie algebra.

Lie bracket: Properties

- $[X, Y] = -[Y, X]$.
- $[X, Y]$ is multi-linear in X, Y.
- It is not associative! That is $[X, [Y, Z]] \neq [[X, Y], Z]$! Indeed, $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (Jacobi's identity).
- Any vector space equipped with such a "bracket" is called a Lie algebra. The space of smooth vector fields is an example.

Lie bracket: Characterisation of coordinate vector fields

Lie bracket: Characterisation of coordinate vector fields

- If X, Y are coordinate vector fields,

Lie bracket: Characterisation of coordinate vector fields

- If X, Y are coordinate vector fields, they Lie-commute (why?).

Lie bracket: Characterisation of coordinate vector fields

- If X, Y are coordinate vector fields, they Lie-commute (why?).
- Conversely, Theorem (proof omitted):

Lie bracket: Characterisation of coordinate vector fields

- If X, Y are coordinate vector fields, they Lie-commute (why?).
- Conversely, Theorem (proof omitted): If $X^1, X^2, \ldots, X^k$ are smooth Lie-commuting vector fields

Lie bracket: Characterisation of coordinate vector fields

- If X, Y are coordinate vector fields, they Lie-commute (why?).
- Conversely, Theorem (proof omitted): If $X^1, X^2, \ldots, X^k$ are smooth Lie-commuting vector fields that are linearly independent at p,

Lie bracket: Characterisation of coordinate vector fields

- If X, Y are coordinate vector fields, they Lie-commute (why?).
- Conversely, Theorem (proof omitted): If $X^1, X^2, \ldots, X^k$ are smooth Lie-commuting vector fields that are linearly independent at p, there is a neighbourhood

Lie bracket: Characterisation of coordinate vector fields

- If X, Y are coordinate vector fields, they Lie-commute (why?).
- Conversely, Theorem (proof omitted): If $X^1, X^2, \ldots, X^k$ are smooth Lie-commuting vector fields that are linearly independent at p, there is a neighbourhood and a coordinate chart

Lie bracket: Characterisation of coordinate vector fields

- If X, Y are coordinate vector fields, they Lie-commute (why?).
- Conversely, Theorem (proof omitted): If $X^1, X^2, \ldots, X^k$ are smooth Lie-commuting vector fields that are linearly independent at p, there is a neighbourhood and a coordinate chart such that $X^i = \frac{\partial}{\partial x^i}$.

Cotangent bundle and one-forms

Cotangent bundle and one-forms

- Recall that T^*M

Cotangent bundle and one-forms

- Recall that T^*M is set theoretically,

Cotangent bundle and one-forms

- Recall that T^*M is set theoretically, $\cup_p T_p^* M$.

Cotangent bundle and one-forms

- Recall that T^*M is set theoretically, $\cup_p T_p^*M$. T^*M is a vector bundle over M.

Cotangent bundle and one-forms

- Recall that T^*M is set theoretically, $\cup_p T_p^*M$. T^*M is a vector bundle over M.
- Given a coordinate chart (U, x),

Cotangent bundle and one-forms

- Recall that T^*M is set theoretically, $\cup_p T_p^*M$. T^*M is a vector bundle over M.
- Given a coordinate chart (U, x), a local smoothly varying basis for T^*M

Cotangent bundle and one-forms

- Recall that T^*M is set theoretically, $\cup_p T_p^*M$. T^*M is a vector bundle over M.
- Given a coordinate chart (U, x), a local smoothly varying basis for T^*M is given by $(\frac{\partial}{\partial x^i})^*$.

Cotangent bundle and one-forms

- Recall that T^*M is set theoretically, $\cup_p T_p^*M$. T^*M is a vector bundle over M.
- Given a coordinate chart (U, x), a local smoothly varying basis for T^*M is given by $(\frac{\partial}{\partial x^i})^*$. A one-form is an element of T_p^*M.

Cotangent bundle and one-forms

- Recall that T^*M is set theoretically, $\cup_p T_p^*M$. T^*M is a vector bundle over M.
- Given a coordinate chart (U, x), a local smoothly varying basis for T^*M is given by $(\frac{\partial}{\partial x^i})^*$. A one-form is an element of T_p^*M. A one-form field

Cotangent bundle and one-forms

- Recall that T^*M is set theoretically, $\cup_p T_p^* M$. T^*M is a vector bundle over M.
- Given a coordinate chart (U, x), a local smoothly varying basis for T^*M is given by $(\frac{\partial}{\partial x^i})^*$. A one-form is an element of $T_p^* M$. A one-form field is a collection of smoothly varying one-forms (

Cotangent bundle and one-forms

- Recall that T^*M is set theoretically, $\cup_p T_p^*M$. T^*M is a vector bundle over M.
- Given a coordinate chart (U, x), a local smoothly varying basis for T^*M is given by $(\frac{\partial}{\partial x^i})^*$. A one-form is an element of T_p^*M. A one-form field is a collection of smoothly varying one-forms (a smooth section of T^*M), i.e.,

Cotangent bundle and one-forms

- Recall that T^*M is set theoretically, $\cup_p T_p^*M$. T^*M is a vector bundle over M.
- Given a coordinate chart (U, x), a local smoothly varying basis for T^*M is given by $(\frac{\partial}{\partial x^i})^*$. A one-form is an element of T_p^*M. A one-form field is a collection of smoothly varying one-forms (a smooth section of T^*M), i.e., $\omega : M \to T^*M$ such that

Cotangent bundle and one-forms

- Recall that T^*M is set theoretically, $\cup_p T_p^*M$. T^*M is a vector bundle over M.
- Given a coordinate chart (U, x), a local smoothly varying basis for T^*M is given by $(\frac{\partial}{\partial x^i})^*$. A one-form is an element of T_p^*M. A one-form field is a collection of smoothly varying one-forms (a smooth section of T^*M), i.e., $\omega : M \to T^*M$ such that $\omega(p) \in T_p^*M$ and

Cotangent bundle and one-forms

- Recall that T^*M is set theoretically, $\cup_p T_p^*M$. T^*M is a vector bundle over M.
- Given a coordinate chart (U, x), a local smoothly varying basis for T^*M is given by $(\frac{\partial}{\partial x^i})^*$. A one-form is an element of T_p^*M. A one-form field is a collection of smoothly varying one-forms (a smooth section of T^*M), i.e., $\omega : M \to T^*M$ such that $\omega(p) \in T_p^*M$ and around every point, there exists a coordinate chart (U, x)

Cotangent bundle and one-forms

- Recall that T^*M is set theoretically, $\cup_p T_p^*M$. T^*M is a vector bundle over M.
- Given a coordinate chart (U, x), a local smoothly varying basis for T^*M is given by $(\frac{\partial}{\partial x^i})^*$. A one-form is an element of T_p^*M. A one-form field is a collection of smoothly varying one-forms (a smooth section of T^*M), i.e., $\omega : M \to T^*M$ such that $\omega(p) \in T_p^*M$ and around every point, there exists a coordinate chart (U, x) such that $\omega = \sum_i \omega_i (\frac{\partial}{\partial x^i})^*$ where the functions ω_i are smooth.

Cotangent bundle and one-forms

- Recall that T^*M is set theoretically, $\cup_p T_p^*M$. T^*M is a vector bundle over M.
- Given a coordinate chart (U, x), a local smoothly varying basis for T^*M is given by $(\frac{\partial}{\partial x^i})^*$. A one-form is an element of T_p^*M. A one-form field is a collection of smoothly varying one-forms (a smooth section of T^*M), i.e., $\omega : M \rightarrow T^*M$ such that $\omega(p) \in T_p^*M$ and around every point, there exists a coordinate chart (U, x) such that $\omega = \sum_i \omega_i (\frac{\partial}{\partial x^i})^*$ where the functions ω_i are smooth.
- As before,

Cotangent bundle and one-forms

- Recall that T^*M is set theoretically, $\cup_p T_p^* M$. T^*M is a vector bundle over M.
- Given a coordinate chart (U, x), a local smoothly varying basis for T^*M is given by $(\frac{\partial}{\partial x^i})^*$. A one-form is an element of $T_p^* M$. A one-form field is a collection of smoothly varying one-forms (a smooth section of T^*M), i.e., $\omega : M \to T^*M$ such that $\omega(p) \in T_p^* M$ and around every point, there exists a coordinate chart (U, x) such that $\omega = \sum_i \omega_i (\frac{\partial}{\partial x^i})^*$ where the functions ω_i are smooth.
- As before, if ω is smooth in one chart,

Cotangent bundle and one-forms

- Recall that T^*M is set theoretically, $\cup_p T^*_p M$. T^*M is a vector bundle over M.
- Given a coordinate chart (U, x), a local smoothly varying basis for T^*M is given by $(\frac{\partial}{\partial x^i})^*$. A one-form is an element of $T^*_p M$. A one-form field is a collection of smoothly varying one-forms (a smooth section of T^*M), i.e., $\omega : M \to T^*M$ such that $\omega(p) \in T^*_p M$ and around every point, there exists a coordinate chart (U, x) such that $\omega = \sum_i \omega_i (\frac{\partial}{\partial x^i})^*$ where the functions ω_i are smooth.
- As before, if ω is smooth in one chart, it is so in all charts:

Cotangent bundle and one-forms

- Recall that T^*M is set theoretically, $\cup_p T_p^*M$. T^*M is a vector bundle over M.
- Given a coordinate chart (U, x), a local smoothly varying basis for T^*M is given by $(\frac{\partial}{\partial x^i})^*$. A one-form is an element of T_p^*M. A one-form field is a collection of smoothly varying one-forms (a smooth section of T^*M), i.e., $\omega : M \to T^*M$ such that $\omega(p) \in T_p^*M$ and around every point, there exists a coordinate chart (U, x) such that $\omega = \sum_i \omega_i (\frac{\partial}{\partial x^i})^*$ where the functions ω_i are smooth.
- As before, if ω is smooth in one chart, it is so in all charts: $\tilde{\omega}_i = \omega_j \frac{\partial x^j}{\partial \tilde{x}^i}$ (why?).

Cotangent bundle and one-forms

- Recall that T^*M is set theoretically, $\cup_p T_p^*M$. T^*M is a vector bundle over M.
- Given a coordinate chart (U, x), a local smoothly varying basis for T^*M is given by $(\frac{\partial}{\partial x^i})^*$. A one-form is an element of T_p^*M. A one-form field is a collection of smoothly varying one-forms (a smooth section of T^*M), i.e., $\omega : M \to T^*M$ such that $\omega(p) \in T_p^*M$ and around every point, there exists a coordinate chart (U, x) such that $\omega = \sum_i \omega_i (\frac{\partial}{\partial x^i})^*$ where the functions ω_i are smooth.
- As before, if ω is smooth in one chart, it is so in all charts: $\tilde{\omega}_i = \omega_j \frac{\partial x^j}{\partial \tilde{x}^i}$ (why?).
- Moreover, given an atlas of U_α of M,

Cotangent bundle and one-forms

- Recall that T^*M is set theoretically, $\cup_p T_p^*M$. T^*M is a vector bundle over M.
- Given a coordinate chart (U, x), a local smoothly varying basis for T^*M is given by $(\frac{\partial}{\partial x^i})^*$. A one-form is an element of T_p^*M. A one-form field is a collection of smoothly varying one-forms (a smooth section of T^*M), i.e., $\omega : M \to T^*M$ such that $\omega(p) \in T_p^*M$ and around every point, there exists a coordinate chart (U, x) such that $\omega = \sum_i \omega_i (\frac{\partial}{\partial x^i})^*$ where the functions ω_i are smooth.
- As before, if ω is smooth in one chart, it is so in all charts: $\tilde{\omega}_i = \omega_j \frac{\partial x^j}{\partial \tilde{x}^i}$ (why?).
- Moreover, given an atlas of U_α of M, and a collection of functions $\omega_{i,\alpha}$ such that

Cotangent bundle and one-forms

- Recall that T^*M is set theoretically, $\cup_p T_p^*M$. T^*M is a vector bundle over M.
- Given a coordinate chart (U, x), a local smoothly varying basis for T^*M is given by $(\frac{\partial}{\partial x^i})^*$. A one-form is an element of T_p^*M. A one-form field is a collection of smoothly varying one-forms (a smooth section of T^*M), i.e., $\omega : M \to T^*M$ such that $\omega(p) \in T_p^*M$ and around every point, there exists a coordinate chart (U, x) such that $\omega = \sum_i \omega_i (\frac{\partial}{\partial x^i})^*$ where the functions ω_i are smooth.
- As before, if ω is smooth in one chart, it is so in all charts: $\tilde{\omega}_i = \omega_j \frac{\partial x^j}{\partial \tilde{x}^i}$ (why?).
- Moreover, given an atlas of U_α of M, and a collection of functions $\omega_{i,\alpha}$ such that on $U_\alpha \cap U_\beta$,

Cotangent bundle and one-forms

- Recall that T^*M is set theoretically, $\cup_p T_p^*M$. T^*M is a vector bundle over M.
- Given a coordinate chart (U, x), a local smoothly varying basis for T^*M is given by $(\frac{\partial}{\partial x^i})^*$. A one-form is an element of T_p^*M. A one-form field is a collection of smoothly varying one-forms (a smooth section of T^*M), i.e., $\omega : M \to T^*M$ such that $\omega(p) \in T_p^*M$ and around every point, there exists a coordinate chart (U, x) such that $\omega = \sum_i \omega_i (\frac{\partial}{\partial x^i})^*$ where the functions ω_i are smooth.
- As before, if ω is smooth in one chart, it is so in all charts: $\tilde{\omega}_i = \omega_j \frac{\partial x^j}{\partial \tilde{x}^i}$ (why?).
- Moreover, given an atlas of U_α of M, and a collection of functions $\omega_{i,\alpha}$ such that on $U_\alpha \cap U_\beta$, $\omega_{i,\alpha} = \omega_{j,\beta} \frac{\partial x^j_\beta}{\partial x\alpha^i}$

Cotangent bundle and one-forms

- Recall that T^*M is set theoretically, $\cup_p T_p^*M$. T^*M is a vector bundle over M.
- Given a coordinate chart (U, x), a local smoothly varying basis for T^*M is given by $(\frac{\partial}{\partial x^i})^*$. A one-form is an element of T_p^*M. A one-form field is a collection of smoothly varying one-forms (a smooth section of T^*M), i.e., $\omega : M \to T^*M$ such that $\omega(p) \in T_p^*M$ and around every point, there exists a coordinate chart (U, x) such that $\omega = \sum_i \omega_i (\frac{\partial}{\partial x^i})^*$ where the functions ω_i are smooth.
- As before, if ω is smooth in one chart, it is so in all charts: $\tilde{\omega}_i = \omega_j \frac{\partial x^j}{\partial \tilde{x}^i}$ (why?).
- Moreover, given an atlas of U_α of M, and a collection of functions $\omega_{i,\alpha}$ such that on $U_\alpha \cap U_\beta$, $\omega_{i,\alpha} = \omega_{j,\beta} \frac{\partial x_\beta^j}{\partial x \alpha^i}$ then there exists a smooth one-form field ω on M

Cotangent bundle and one-forms

- Recall that T^*M is set theoretically, $\cup_p T_p^*M$. T^*M is a vector bundle over M.
- Given a coordinate chart (U, x), a local smoothly varying basis for T^*M is given by $(\frac{\partial}{\partial x^i})^*$. A one-form is an element of T_p^*M. A one-form field is a collection of smoothly varying one-forms (a smooth section of T^*M), i.e., $\omega : M \to T^*M$ such that $\omega(p) \in T_p^*M$ and around every point, there exists a coordinate chart (U, x) such that $\omega = \sum_i \omega_i (\frac{\partial}{\partial x^i})^*$ where the functions ω_i are smooth.
- As before, if ω is smooth in one chart, it is so in all charts: $\tilde{\omega}_i = \omega_j \frac{\partial x^j}{\partial \tilde{x}^i}$ (why?).
- Moreover, given an atlas of U_α of M, and a collection of functions $\omega_{i,\alpha}$ such that on $U_\alpha \cap U_\beta$, $\omega_{i,\alpha} = \omega_{j,\beta} \frac{\partial x_\beta^j}{\partial x \alpha^i}$ then there exists a smooth one-form field ω on M whose coordinate representations are given by $\omega_{i,\alpha}$ (why?)

Differential of a function

Differential of a function

- Given a smooth function $f : M \to \mathbb{R}$,

Differential of a function

- Given a smooth function $f : M \to \mathbb{R}$, what is its derivative?

Differential of a function

- Given a smooth function $f : M \to \mathbb{R}$, what is its derivative?
- Sure, one way to answer the question

Differential of a function

- Given a smooth function $f : M \to \mathbb{R}$, what is its derivative?
- Sure, one way to answer the question is to say it is f_*.

Differential of a function

- Given a smooth function $f : M \to \mathbb{R}$, what is its derivative?
- Sure, one way to answer the question is to say it is f_*. However, f_* can at best

Differential of a function

- Given a smooth function $f : M \to \mathbb{R}$, what is its derivative?
- Sure, one way to answer the question is to say it is f_*. However, f_* can at best be thought of as a function $f_* : TM \to T\mathbb{R}$.

Differential of a function

- Given a smooth function $f : M \to \mathbb{R}$, what is its derivative?
- Sure, one way to answer the question is to say it is f_*. However, f_* can at best be thought of as a function $f_* : TM \to T\mathbb{R}$. On the other hand,

Differential of a function

- Given a smooth function $f : M \to \mathbb{R}$, what is its derivative?
- Sure, one way to answer the question is to say it is f_*. However, f_* can at best be thought of as a function $f_* : TM \to T\mathbb{R}$. On the other hand, if $M = \mathbb{R}^n$,

Differential of a function

- Given a smooth function $f : M \to \mathbb{R}$, what is its derivative?
- Sure, one way to answer the question is to say it is f_*. However, f_* can at best be thought of as a function $f_* : TM \to T\mathbb{R}$. On the other hand, if $M = \mathbb{R}^n$, we can think of the derivative as $Df : \mathbb{R}^n \to \mathbb{R}^n$.

Differential of a function

- Given a smooth function $f : M \to \mathbb{R}$, what is its derivative?
- Sure, one way to answer the question is to say it is f_*. However, f_* can at best be thought of as a function $f_* : TM \to T\mathbb{R}$. On the other hand, if $M = \mathbb{R}^n$, we can think of the derivative as $Df : \mathbb{R}^n \to \mathbb{R}^n$.
- So is there a function $df : M \to$ *something* ?

Differential of a function

- Given a smooth function $f : M \to \mathbb{R}$, what is its derivative?
- Sure, one way to answer the question is to say it is f_*. However, f_* can at best be thought of as a function $f_* : TM \to T\mathbb{R}$. On the other hand, if $M = \mathbb{R}^n$, we can think of the derivative as $Df : \mathbb{R}^n \to \mathbb{R}^n$.
- So is there a function $df : M \to$ *something* ?
- The most naive way to define it is

Differential of a function

- Given a smooth function $f : M \to \mathbb{R}$, what is its derivative?
- Sure, one way to answer the question is to say it is f_*. However, f_* can at best be thought of as a function $f_* : TM \to T\mathbb{R}$. On the other hand, if $M = \mathbb{R}^n$, we can think of the derivative as $Df : \mathbb{R}^n \to \mathbb{R}^n$.
- So is there a function $df : M \to$ *something* ?
- The most naive way to define it is take a chart (U, x) and take $(\frac{\partial f}{\partial x^1}, \dots)$.

Differential of a function

- Given a smooth function $f : M \to \mathbb{R}$, what is its derivative?
- Sure, one way to answer the question is to say it is f_*. However, f_* can at best be thought of as a function $f_* : TM \to T\mathbb{R}$. On the other hand, if $M = \mathbb{R}^n$, we can think of the derivative as $Df : \mathbb{R}^n \to \mathbb{R}^n$.
- So is there a function $df : M \to$ *something* ?
- The most naive way to define it is take a chart (U, x) and take $(\frac{\partial f}{\partial x^1}, \ldots)$. However, when we change charts,

Differential of a function

- Given a smooth function $f : M \to \mathbb{R}$, what is its derivative?
- Sure, one way to answer the question is to say it is f_*. However, f_* can at best be thought of as a function $f_* : TM \to T\mathbb{R}$. On the other hand, if $M = \mathbb{R}^n$, we can think of the derivative as $Df : \mathbb{R}^n \to \mathbb{R}^n$.
- So is there a function $df : M \to$ *something* ?
- The most naive way to define it is take a chart (U, x) and take $(\frac{\partial f}{\partial x^1}, \ldots)$. However, when we change charts, these vectors change to

Differential of a function

- Given a smooth function $f : M \to \mathbb{R}$, what is its derivative?
- Sure, one way to answer the question is to say it is f_*. However, f_* can at best be thought of as a function $f_* : TM \to T\mathbb{R}$. On the other hand, if $M = \mathbb{R}^n$, we can think of the derivative as $Df : \mathbb{R}^n \to \mathbb{R}^n$.
- So is there a function $df : M \to$ *something* ?
- The most naive way to define it is take a chart (U, x) and take $(\frac{\partial f}{\partial x^1}, \ldots)$. However, when we change charts, these vectors change to $(\frac{\partial f}{\partial x^i} \frac{\partial x^i}{\partial \tilde{x}^1}, \ldots)$ which is

Differential of a function

- Given a smooth function $f : M \to \mathbb{R}$, what is its derivative?
- Sure, one way to answer the question is to say it is f_*. However, f_* can at best be thought of as a function $f_* : TM \to T\mathbb{R}$. On the other hand, if $M = \mathbb{R}^n$, we can think of the derivative as $Df : \mathbb{R}^n \to \mathbb{R}^n$.
- So is there a function $df : M \to$ *something* ?
- The most naive way to define it is take a chart (U, x) and take $(\frac{\partial f}{\partial x^1}, \ldots)$. However, when we change charts, these vectors change to $(\frac{\partial f}{\partial x^i} \frac{\partial x^i}{\partial \tilde{x}^1}, \ldots)$ which is exactly the way one-form fields change!

Differential of a function

- Given a smooth function $f : M \to \mathbb{R}$, what is its derivative?
- Sure, one way to answer the question is to say it is f_*. However, f_* can at best be thought of as a function $f_* : TM \to T\mathbb{R}$. On the other hand, if $M = \mathbb{R}^n$, we can think of the derivative as $Df : \mathbb{R}^n \to \mathbb{R}^n$.
- So is there a function $df : M \to$ *something* ?
- The most naive way to define it is take a chart (U, x) and take $(\frac{\partial f}{\partial x^1}, \ldots)$. However, when we change charts, these vectors change to $(\frac{\partial f}{\partial x^i}\frac{\partial x^i}{\partial \tilde{x}^1}, \ldots)$ which is exactly the way one-form fields change!
- Thus df must be thought of as a one-form field!

Differential of a function

Differential of a function

- Invariantly speaking,

Differential of a function

- Invariantly speaking, $df(X_p) := X_p(f)$.

Differential of a function

- Invariantly speaking, $df(X_p) := X_p(f)$.
- Suppose x^i are coordinates,

Differential of a function

- Invariantly speaking, $df(X_p) := X_p(f)$.
- Suppose x^i are coordinates, then $dx^i(\frac{\partial}{\partial x^j}) = \delta^i_j$, i.e., $dx^i = (\frac{\partial}{\partial x^i})^*$.

Differential of a function

- Invariantly speaking, $df(X_p) := X_p(f)$.
- Suppose x^i are coordinates, then $dx^i(\frac{\partial}{\partial x^j}) = \delta^i_j$, i.e., $dx^i = (\frac{\partial}{\partial x^i})^*$. Thus any one-form field ω

Differential of a function

- Invariantly speaking, $df(X_p) := X_p(f)$.
- Suppose x^i are coordinates, then $dx^i(\frac{\partial}{\partial x^j}) = \delta^i_j$, i.e., $dx^i = (\frac{\partial}{\partial x^i})^*$. Thus any one-form field ω is $\omega = \omega_i dx^i$.

Differential of a function

- Invariantly speaking, $df(X_p) := X_p(f)$.
- Suppose x^i are coordinates, then $dx^i(\frac{\partial}{\partial x^j}) = \delta^i_j$, i.e., $dx^i = (\frac{\partial}{\partial x^i})^*$. Thus any one-form field ω is $\omega = \omega_i dx^i$.
- Suppose γ is a smooth path,

Differential of a function

- Invariantly speaking, $df(X_p) := X_p(f)$.
- Suppose x^i are coordinates, then $dx^i(\frac{\partial}{\partial x^j}) = \delta^i_j$, i.e., $dx^i = (\frac{\partial}{\partial x^i})^*$. Thus any one-form field ω is $\omega = \omega_i dx^i$.
- Suppose γ is a smooth path, then $df(\gamma') = \gamma'(f) = (f \circ \gamma)'$.

Differential of a function

- Invariantly speaking, $df(X_p) := X_p(f)$.
- Suppose x^i are coordinates, then $dx^i(\frac{\partial}{\partial x^j}) = \delta^i_j$, i.e., $dx^i = (\frac{\partial}{\partial x^i})^*$. Thus any one-form field ω is $\omega = \omega_i dx^i$.
- Suppose γ is a smooth path, then $df(\gamma') = \gamma'(f) = (f \circ \gamma)'$. Thus in some sense,

Differential of a function

- Invariantly speaking, $df(X_p) := X_p(f)$.
- Suppose x^i are coordinates, then $dx^i(\frac{\partial}{\partial x^j}) = \delta^i_j$, i.e., $dx^i = (\frac{\partial}{\partial x^i})^*$. Thus any one-form field ω is $\omega = \omega_i dx^i$.
- Suppose γ is a smooth path, then $df(\gamma') = \gamma'(f) = (f \circ \gamma)'$. Thus in some sense, df is the "right" analogue of

Differential of a function

- Invariantly speaking, $df(X_p) := X_p(f)$.
- Suppose x^i are coordinates, then $dx^i(\frac{\partial}{\partial x^j}) = \delta^i_j$, i.e., $dx^i = (\frac{\partial}{\partial x^i})^*$. Thus any one-form field ω is $\omega = \omega_i dx^i$.
- Suppose γ is a smooth path, then $df(\gamma') = \gamma'(f) = (f \circ \gamma)'$. Thus in some sense, df is the “right” analogue of Newton's infinitesimals.

Differential of a function

- Invariantly speaking, $df(X_p) := X_p(f)$.
- Suppose x^i are coordinates, then $dx^i(\frac{\partial}{\partial x^j}) = \delta^i_j$, i.e., $dx^i = (\frac{\partial}{\partial x^i})^*$. Thus any one-form field ω is $\omega = \omega_i dx^i$.
- Suppose γ is a smooth path, then $df(\gamma') = \gamma'(f) = (f \circ \gamma)'$. Thus in some sense, df is the "right" analogue of Newton's infinitesimals.
- $d(fg) = dfg + fdg, d(f+g) = df + dg, d(f/g) = \frac{dfg - fdg}{g^2}, d(c) = 0.$

Differential of a function

- Invariantly speaking, $df(X_p) := X_p(f)$.
- Suppose x^i are coordinates, then $dx^i(\frac{\partial}{\partial x^j}) = \delta^i_j$, i.e., $dx^i = (\frac{\partial}{\partial x^i})^*$. Thus any one-form field ω is $\omega = \omega_i dx^i$.
- Suppose γ is a smooth path, then $df(\gamma') = \gamma'(f) = (f \circ \gamma)'$. Thus in some sense, df is the "right" analogue of Newton's infinitesimals.
- $d(fg) = dfg + fdg, d(f+g) = df + dg, d(f/g) = \frac{dfg - fdg}{g^2}, d(c) = 0$. Moreover, if $df \equiv 0$ on a connected manifold,

Differential of a function

- Invariantly speaking, $df(X_p) := X_p(f)$.
- Suppose x^i are coordinates, then $dx^i(\frac{\partial}{\partial x^j}) = \delta^i_j$, i.e., $dx^i = (\frac{\partial}{\partial x^i})^*$. Thus any one-form field ω is $\omega = \omega_i dx^i$.
- Suppose γ is a smooth path, then $df(\gamma') = \gamma'(f) = (f \circ \gamma)'$. Thus in some sense, df is the "right" analogue of Newton's infinitesimals.
- $d(fg) = dfg + fdg, d(f+g) = df + dg, d(f/g) = \frac{dfg - fdg}{g^2}, d(c) = 0$. Moreover, if $df \equiv 0$ on a connected manifold, f is constant on the manifold:

Differential of a function

- Invariantly speaking, $df(X_p) := X_p(f)$.
- Suppose x^i are coordinates, then $dx^i(\frac{\partial}{\partial x^j}) = \delta^i_j$, i.e., $dx^i = (\frac{\partial}{\partial x^i})^*$. Thus any one-form field ω is $\omega = \omega_i dx^i$.
- Suppose γ is a smooth path, then $df(\gamma') = \gamma'(f) = (f \circ \gamma)'$. Thus in some sense, df is the "right" analogue of Newton's infinitesimals.
- $d(fg) = dfg + fdg, d(f+g) = df + dg, d(f/g) = \frac{dfg - fdg}{g^2}, d(c) = 0$. Moreover, if $df \equiv 0$ on a connected manifold, f is constant on the manifold: Indeed, let $f(p) = c$ for some p.

Differential of a function

- Invariantly speaking, $df(X_p) := X_p(f)$.
- Suppose x^i are coordinates, then $dx^i(\frac{\partial}{\partial x^j}) = \delta^i_j$, i.e., $dx^i = (\frac{\partial}{\partial x^i})^*$. Thus any one-form field ω is $\omega = \omega_i dx^i$.
- Suppose γ is a smooth path, then $df(\gamma') = \gamma'(f) = (f \circ \gamma)'$. Thus in some sense, *df* is the "right" analogue of Newton's infinitesimals.
- $d(fg) = dfg + fdg, d(f+g) = df + dg, d(f/g) = \frac{dfg - fdg}{g^2}, d(c) = 0$. Moreover, if $df \equiv 0$ on a connected manifold, f is constant on the manifold: Indeed, let $f(p) = c$ for some p. Then the set of all $q \in M$ such that $f(q) = p$ is non-empty and closed.

Differential of a function

- Invariantly speaking, $df(X_p) := X_p(f)$.
- Suppose x^i are coordinates, then $dx^i(\frac{\partial}{\partial x^j}) = \delta^i_j$, i.e., $dx^i = (\frac{\partial}{\partial x^i})^*$. Thus any one-form field ω is $\omega = \omega_i dx^i$.
- Suppose γ is a smooth path, then $df(\gamma') = \gamma'(f) = (f \circ \gamma)'$. Thus in some sense, df is the "right" analogue of Newton's infinitesimals.
- $d(fg) = dfg + fdg, d(f+g) = df + dg, d(f/g) = \frac{dfg - fdg}{g^2}, d(c) = 0$. Moreover, if $df \equiv 0$ on a connected manifold, f is constant on the manifold: Indeed, let $f(p) = c$ for some p. Then the set of all $q \in M$ such that $f(q) = p$ is non-empty and closed. It is also open:

Differential of a function

- Invariantly speaking, $df(X_p) := X_p(f)$.
- Suppose x^i are coordinates, then $dx^i(\frac{\partial}{\partial x^j}) = \delta^i_j$, i.e., $dx^i = (\frac{\partial}{\partial x^i})^*$. Thus any one-form field ω is $\omega = \omega_i dx^i$.
- Suppose γ is a smooth path, then $df(\gamma') = \gamma'(f) = (f \circ \gamma)'$. Thus in some sense, df is the "right" analogue of Newton's infinitesimals.
- $d(fg) = dfg + fdg, d(f+g) = df + dg, d(f/g) = \frac{dfg - fdg}{g^2}, d(c) = 0$. Moreover, if $df \equiv 0$ on a connected manifold, f is constant on the manifold: Indeed, let $f(p) = c$ for some p. Then the set of all $q \in M$ such that $f(q) = p$ is non-empty and closed. It is also open: on a coordinate neighbourhood of q, $df = 0$ iff $\frac{\partial f}{\partial x^i} = 0$ and hence $f = c$ on that neighbourhood.

Differential of a function

- Invariantly speaking, $df(X_p) := X_p(f)$.
- Suppose x^i are coordinates, then $dx^i(\frac{\partial}{\partial x^j}) = \delta^i_j$, i.e., $dx^i = (\frac{\partial}{\partial x^i})^*$. Thus any one-form field ω is $\omega = \omega_i dx^i$.
- Suppose γ is a smooth path, then $df(\gamma') = \gamma'(f) = (f \circ \gamma)'$. Thus in some sense, df is the "right" analogue of Newton's infinitesimals.
- $d(fg) = dfg + fdg, d(f+g) = df + dg, d(f/g) = \frac{dfg - fdg}{g^2}, d(c) = 0$. Moreover, if $df \equiv 0$ on a connected manifold, f is constant on the manifold: Indeed, let $f(p) = c$ for some p. Then the set of all $q \in M$ such that $f(q) = p$ is non-empty and closed. It is also open: on a coordinate neighbourhood of q, $df = 0$ iff $\frac{\partial f}{\partial x^i} = 0$ and hence $f = c$ on that neighbourhood. By connectedness we are done. □

Pullback

Pullback

- Just as we can pushforward tangent vectors $(F_*)_p : T_pM \to T_{F(p)}N$,

Pullback

- Just as we can pushforward tangent vectors $(F_*)_p : T_pM \to T_{F(p)}N$, the dual map can be used to pullback cotangent vectors/one-forms:

Pullback

- Just as we can pushforward tangent vectors $(F_*)_p : T_pM \to T_{F(p)}N$, the dual map can be used to pullback cotangent vectors/one-forms: $(F^*)_{F(p)} : T^*_{F(p)}N \to T^*_pM$ given by

Pullback

- Just as we can pushforward tangent vectors $(F_*)_p : T_pM \to T_{F(p)}N$, the dual map can be used to pullback cotangent vectors/one-forms: $(F^*)_{F(p)} : T^*_{F(p)}N \to T^*_pM$ given by $(F^*)_{F(p)}(\omega_{F(p)})(X_p) = \omega_{F(p)}((F_*)_pX_p)$.

Pullback

- Just as we can pushforward tangent vectors $(F_*)_p : T_pM \to T_{F(p)}N$, the dual map can be used to pullback cotangent vectors/one-forms: $(F^*)_{F(p)} : T^*_{F(p)}N \to T^*_pM$ given by $(F^*)_{F(p)}(\omega_{F(p)})(X_p) = \omega_{F(p)}((F_*)_pX_p)$.
- In fact,

Pullback

- Just as we can pushforward tangent vectors $(F_*)_p : T_pM \to T_{F(p)}N$, the dual map can be used to pullback cotangent vectors/one-forms: $(F^*)_{F(p)} : T^*_{F(p)}N \to T^*_pM$ given by $(F^*)_{F(p)}(\omega_{F(p)})(X_p) = \omega_{F(p)}((F_*)_pX_p)$.
- In fact, while we cannot pushforward vector fields,

Pullback

- Just as we can pushforward tangent vectors $(F_*)_p : T_pM \to T_{F(p)}N$, the dual map can be used to pullback cotangent vectors/one-forms: $(F^*)_{F(p)} : T^*_{F(p)}N \to T^*_pM$ given by $(F^*)_{F(p)}(\omega_{F(p)})(X_p) = \omega_{F(p)}((F_*)_pX_p)$.
- In fact, while we cannot pushforward vector fields, we can always pullback one-form fields:

Pullback

- Just as we can pushforward tangent vectors $(F_*)_p : T_pM \to T_{F(p)}N$, the dual map can be used to pullback cotangent vectors/one-forms: $(F^*)_{F(p)} : T^*_{F(p)}N \to T^*_pM$ given by $(F^*)_{F(p)}(\omega_{F(p)})(X_p) = \omega_{F(p)}((F_*)_pX_p)$.
- In fact, while we cannot pushforward vector fields, we can always pullback one-form fields: $(F^*)\omega(p)(X_p) = \omega_{F(p)}((F_*)_pX_p)$.

Pullback

- Just as we can pushforward tangent vectors $(F_*)_p : T_pM \to T_{F(p)}N$, the dual map can be used to pullback cotangent vectors/one-forms: $(F^*)_{F(p)} : T^*_{F(p)}N \to T^*_pM$ given by $(F^*)_{F(p)}(\omega_{F(p)})(X_p) = \omega_{F(p)}((F_*)_pX_p)$.
- In fact, while we cannot pushforward vector fields, we can always pullback one-form fields: $(F^*)\omega(p)(X_p) = \omega_{F(p)}((F_*)_pX_p)$.
- In coordinates,

Pullback

- Just as we can pushforward tangent vectors $(F_*)_p : T_pM \to T_{F(p)}N$, the dual map can be used to pullback cotangent vectors/one-forms: $(F^*)_{F(p)} : T^*_{F(p)}N \to T^*_pM$ given by $(F^*)_{F(p)}(\omega_{F(p)})(X_p) = \omega_{F(p)}((F_*)_p X_p)$.
- In fact, while we cannot pushforward vector fields, we can always pullback one-form fields: $(F^*)\omega(p)(X_p) = \omega_{F(p)}((F_*)_p X_p)$.
- In coordinates, $F*(dx^i)(\frac{\partial}{\partial x^j}) = dx^i(\frac{\partial F^k}{\partial x^j}\frac{\partial}{\partial x^k}) = \frac{\partial F^i}{\partial x^j}$, i.e, $F^*(dx^i) = dF^i = dF^i$.

Pullback

- Just as we can pushforward tangent vectors $(F_*)_p : T_pM \to T_{F(p)}N$, the dual map can be used to pullback cotangent vectors/one-forms: $(F^*)_{F(p)} : T^*_{F(p)}N \to T^*_pM$ given by $(F^*)_{F(p)}(\omega_{F(p)})(X_p) = \omega_{F(p)}((F_*)_pX_p)$.
- In fact, while we cannot pushforward vector fields, we can always pullback one-form fields: $(F^*)\omega(p)(X_p) = \omega_{F(p)}((F_*)_pX_p)$.
- In coordinates, $F * (dx^i)(\frac{\partial}{\partial x^j}) = dx^i(\frac{\partial F^k}{\partial x^j}\frac{\partial}{\partial x^k}) = \frac{\partial F^i}{\partial x^j}$, i.e, $F^*(dx^i) = dF^i = dF^i$.
- For ease of notation,

Pullback

- Just as we can pushforward tangent vectors $(F_*)_p : T_pM \to T_{F(p)}N$, the dual map can be used to pullback cotangent vectors/one-forms: $(F^*)_{F(p)} : T^*_{F(p)}N \to T^*_pM$ given by $(F^*)_{F(p)}(\omega_{F(p)})(X_p) = \omega_{F(p)}((F_*)_pX_p)$.
- In fact, while we cannot pushforward vector fields, we can always pullback one-form fields: $(F^*)\omega(p)(X_p) = \omega_{F(p)}((F_*)_pX_p)$.
- In coordinates, $F*(dx^i)(\frac{\partial}{\partial x^j}) = dx^i(\frac{\partial F^k}{\partial x^j}\frac{\partial}{\partial x^k}) = \frac{\partial F^i}{\partial x^j}$, i.e, $F^*(dx^i) = dF^i = dF^i$.
- For ease of notation, if we denote $F^*f = f \circ F$,

Pullback

- Just as we can pushforward tangent vectors $(F_*)_p : T_pM \to T_{F(p)}N$, the dual map can be used to pullback cotangent vectors/one-forms: $(F^*)_{F(p)} : T^*_{F(p)}N \to T^*_pM$ given by $(F^*)_{F(p)}(\omega_{F(p)})(X_p) = \omega_{F(p)}((F_*)_pX_p)$.
- In fact, while we cannot pushforward vector fields, we can always pullback one-form fields: $(F^*)\omega(p)(X_p) = \omega_{F(p)}((F_*)_pX_p)$.
- In coordinates, $F * (dx^i)(\frac{\partial}{\partial x^j}) = dx^i(\frac{\partial F^k}{\partial x^j}\frac{\partial}{\partial x^k}) = \frac{\partial F^i}{\partial x^j}$, i.e, $F^*(dx^i) = dF^i = dF^i$.
- For ease of notation, if we denote $F^*f = f \circ F$, then $F^*(\omega_i dx^i) = \omega_i \circ F dF^i = F^*\omega_i dF^*x^i$.

Pullback

- Just as we can pushforward tangent vectors $(F_*)_p : T_pM \to T_{F(p)}N$, the dual map can be used to pullback cotangent vectors/one-forms: $(F^*)_{F(p)} : T^*_{F(p)}N \to T^*_pM$ given by $(F^*)_{F(p)}(\omega_{F(p)})(X_p) = \omega_{F(p)}((F_*)_pX_p)$.
- In fact, while we cannot pushforward vector fields, we can always pullback one-form fields: $(F^*)\omega(p)(X_p) = \omega_{F(p)}((F_*)_pX_p)$.
- In coordinates, $F*(dx^i)(\frac{\partial}{\partial x^j}) = dx^i(\frac{\partial F^k}{\partial x^j}\frac{\partial}{\partial x^k}) = \frac{\partial F^i}{\partial x^j}$, i.e, $F^*(dx^i) = dF^i = dF^i$.
- For ease of notation, if we denote $F^*f = f \circ F$, then $F^*(\omega_i dx^i) = \omega_i \circ F dF^i = F^*\omega_i dF^*x^i$.
- As an example,

Pullback

- Just as we can pushforward tangent vectors $(F_*)_p : T_pM \to T_{F(p)}N$, the dual map can be used to pullback cotangent vectors/one-forms: $(F^*)_{F(p)} : T^*_{F(p)}N \to T^*_pM$ given by $(F^*)_{F(p)}(\omega_{F(p)})(X_p) = \omega_{F(p)}((F_*)_pX_p)$.
- In fact, while we cannot pushforward vector fields, we can always pullback one-form fields: $(F^*)\omega(p)(X_p) = \omega_{F(p)}((F_*)_pX_p)$.
- In coordinates, $F*(dx^i)(\frac{\partial}{\partial x^j}) = dx^i(\frac{\partial F^k}{\partial x^j}\frac{\partial}{\partial x^k}) = \frac{\partial F^i}{\partial x^j}$, i.e, $F^*(dx^i) = dF^i = dF^i$.
- For ease of notation, if we denote $F^*f = f \circ F$, then $F^*(\omega_i dx^i) = \omega_i \circ F dF^i = F^*\omega_i dF^*x^i$.
- As an example, if $y = F(x) = x^2$ and $\omega = 3y^4dy$, then $F*\omega = 3(x^2)^4 2xdx$.