MA 229/MA 235 - Lecture 14

IISc

Recap

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- Slice charts for embedded submanifolds.

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- Using these results we can show that submanifolds have a unique smooth structure.

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- Sard's theorem (a weak version): For a smooth map $F : M \to N$, the set of regular values is *dense* in N.
- In particular, if $f : M \to \mathbb{R}$ is a smooth exhaustion, then there is an increasing sequence $c_i \to \infty$ such that $f^{-1}(c_i)$ is a smooth manifold and $f^{-1}(-\infty, c_i]$ form an exhaustion. In fact, $f^{-1}(-\infty, c_i]$ form manifolds-with-boundary (why?).
- There cannot be an onto smooth map from $\mathbb{R}$ to $\mathbb{R}^2$: Indeed, if there is such a map, then there is a $c \in \mathbb{R}^2$ such that $f^{-1}(c)$ is regular level set. Hence $f^{-1}(c)$ is a submanifold of dimension $1 - 2 = -1$! A contradiction. On the other hand, there are continuous space filling curves.