MA 229/MA 235 - Lecture 20

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Recap

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- One form fields, differential of a function, and pullback.

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- Proof: This set is linearly independent: Indeed, if $c_{i_1 i_2 \ldots}(e_1^{i_1}) \otimes (e_2^{i_2}) \ldots = 0$, then acting on $(e_{1,j_1}, e_{2,j_2}, \ldots)$ we get $c_{j_1 j_2 \ldots} = 0$. This is true for all $j_1, j_2 \ldots$. Hence we are done. It spans the space:

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- Proof: Suppose (V', π') is another such space. Then consider the map $\tilde{\pi}' : V_1 \otimes V_2 \to V'$ induced from π'.

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