MA 229/MA 235 - Lecture 21

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Recap

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- Tensor products.

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- Types of tensors, symmetric and alternating tensors (forms). Symmetrisation, Anti-symmetrisation.

Tensor bundles and tensor fields

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