MA 229/MA 235 - Lecture 8

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Recap

- More examples of smooth manifolds.

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- Manifolds-with-boundary.

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- Manifolds-with-boundary.
- Smooth maps.

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- Theorem: Suppose M is a smooth manifold with or without boundary. Let χ be an open cover of M. Then there exists a smooth partition of unity subordinate to it. There also exists a partition-of-unity consisting of compact supports subordinate to a locally finite countable open refinement.