MA 229/MA 235 - Lecture 24

IISc

Recap

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- Differential forms bundle.

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- Exterior derivative and its properties.

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- Closed and exact forms.

Integration in $\mathbb{R}^n$

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