MA 229/MA 235 - Lecture 10

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Recap

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- Applications:

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- Derivations on $\mathbb{R}^n$ and isomorphism using directional derivatives.

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- Proof: 1-1: If $(i_*)_p(v) = 0$, then whenever $f \in C^\infty(M)$, and $v(f\|_U) = 0$, then suppose $g \in C^\infty(U)$. Let $\rho : M \to \mathbb{R}$ be a bump function equal to 1 in a neighbourhood of p and $supp(\rho) \subset U$. Thus $\rho g : M \to \mathbb{R}$ agrees with f in a neighbourhood of p.

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- Corollary: The dimension of T_pM even for manifolds-with-boundary is $dim(M)$.

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- Let $M_1, M_2, \ldots, M_k$ be smooth manifolds (without boundary). Then $\alpha_p : T_p(M_1 \times M_2 \ldots) \to T_pM_1 \times T_pM_2 \ldots$ given by $\alpha_p(v) = ((\pi_1)_*(v), (\pi_2)_*(v), \ldots)$ is an isomorphism.

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