MA 229/MA 235 - Lecture 16

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Recap

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- Defined smooth vector fields and gave examples.

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- Defined smooth vector fields and gave examples.
- Defined the tangent bundle and proved that smooth vector fields are vector fields that are also smooth maps.

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- More generally, what does it mean to have a family of smoothly varying vector spaces?

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- Conversely, given such a collection of smooth sections, the map $L : U \times \mathbb{R}^r \to V$ given by

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- The part about local triviality implies the following: There is a smooth collection of functions $s_1, \ldots, s_r : U \to V$ such that $s_i(q) \in V_q \;\forall\; q \in U$ and $s_1(q), \ldots, s_r(q)$ forms a basis for V_q (there is a local smoothly varying basis). Indeed, if local triviality holds, simply take $s_i = T^{-1}(e_i)$.
- This motivates the following definition: Let V be a smooth vector bundle over a smooth manifold M. A smooth map $s : M \to V$ such that $s(p) \in V_p$, i.e., $\pi(s(p)) = p$ is called a smooth section. Smooth vector fields are smooth sections of TM.
- What we said above is that a local trivialisation gives a collection of smooth sections s_i such that $s_i(p)$ is a basis of V_p.
- Conversely, given such a collection of smooth sections, the map $L : U \times \mathbb{R}^r \to V$ given by $L(p, \vec{v}) = v^i s_i(p)$ is a smooth $1-1$ map such that L^{-1} is a local trivialisation (HW).

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- Def: Let $S \subset V$ be a vector bundle such that the inclusion map is an embedding and a $1-1$ vector bundle map. Then S is said to be a subbundle of V.

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