MA 229/MA 235 - Lecture 26

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Recap

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- Orientation through top forms and charts.

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- Examples (including hypersurfaces).

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- The problem is that we have to use a partition-of-unity and such things are practically impossible to integrate explicitly!
- If we were to do it naively, we would have simply done $\int_{x^2+y^2\leq 1} x^2 dxdy = \int_0^{2\pi} \int_0^1 r^2 \cos^2(\theta) r dr d\theta$.