MA 229/MA 235 - Lecture 17

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Recap

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- Defined the dual bundle and constructed the cotangent bundle as a special case.

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- That theorem is proven by rewriting the system as an integral equation and using an iterative method and the contraction mapping principle. For uniqueness and smoothness, one needs to put in more effort (Gronwall's inequality).
- The ϵ_p can be finite and U_p need not be all of M.

New integral curves from old ones

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Flows of compactly supported vector fields

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- Proof: Let p be a starting point. Let T be the supremum of all ϵ such that the integral curve exists on $(-\epsilon, \epsilon)$. If $T < \infty$, then firstly p is within the support of X (why?). Secondly, consider a sequence $t_n \to T$.

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Flows of compactly supported vector fields

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- Theorem: If X is compactly supported, then $\theta : \mathbb{R} \times M$ given by $\theta_t(p) = \gamma(t)$ where γ is the integral curve of X starting from p (this map is called the time-t flow of X) is smooth and

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- Theorem: If X is compactly supported, then $\theta : \mathbb{R} \times M$ given by $\theta_t(p) = \gamma(t)$ where γ is the integral curve of X starting from p (this map is called the time-t flow of X) is smooth and a diffeomorphism for each fixed t.

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