HW 3 (to be submitted by Feb)

1. (5 marks) Let $U \subset \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}^k$ (where $k \leq n$) be a smooth function. Suppose Df_a has rank k whenever $f(a) = 0$. Prove that $f^{-1}(0)$ can be made into a smooth manifold of dimension $n - k$.

2. (15 marks)

   (a) (5 marks) Consider the complex projective space $\mathbb{CP}^n = (\mathbb{C}^{n+1} - \{0\})/(X \sim \lambda X \mid \lambda \in \mathbb{C} - \{0\}$. Give it a $2n$-dimensional smooth manifold structure akin to that of $\mathbb{RP}^n$.

   (b) (5 marks) Prove that $(S^{2n+1} \subset \mathbb{C}^{n+1})/(p \sim e^{i\theta} p)$ can be made into a $2n$-dimensional smooth manifold such that the projection map from S^{2n+1} is smooth.

   (c) (5 marks) Prove that $\mathbb{CP}^n$ is diffeomorphic to $S^{2n+1}/(p \sim e^{i\theta} p)$. (Note that technically this problem is not well-defined because if you come up with crazy smooth structures on both spaces, then they are not diffeomorphic, but I highly doubt you can actually come up with non-diffeomorphic smooth structures!)

3. (5 marks) Let M be a smooth non-empty n-manifold ($n \geq 1$) with or without boundary. Prove that the space of smooth functions $f : M \to \mathbb{R}$ is infinite-dimensional.