Crash Course on Morse Theory

Ref: Nicolaescu - An Invitation to Morse Theory

Fix a closed Riemannian mfld (X, g)

Defn: $f : X \to \mathbb{R}$ is morse if the follo. holds:

If $\nabla f(p) = 0$ (i.e., p is a critical pt.)

then $Hess_p(f) = \nabla^2 f$

$= \text{"} (\partial_{ij} f)_{ij} \text{"}$

has $\det(Hess_p(f)) \neq 0$

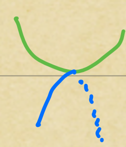
Aim: Extract invariants of X from the fn. $f : X \to \mathbb{R}$

Warm-up: Euler char

Claim: $\chi(X) =$ "signed" $\# Crit(f)$

What are the signs?

Let λ_i be the eigenvalues of $Hess_p(f)$



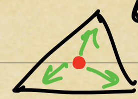
Saddle

$\Rightarrow$ index at $p := i(p) = \#$ -ve λ_i

"# of direction f decreases along"

Claim: $\chi(X) = \sum_{p \in \mathrm{Crit}(f)} (-1)^{i(p)}$

Sketch of proof: Triangulate your mfld

 $\leftsquigarrow$ k-cell

$\Rightarrow$ every k-cell $\leftrightsquigarrow$ $p \in \mathrm{Crit}(f)$ w/ $i(p) = k$

$$\Rightarrow \sum_{p \in \cdots} (-1)^{i(p)} = \sum_k (\# k\text{cells})(-1)^k = \chi(X) \qquad \blacksquare$$

More Refined Invariants: Homology

Heuristic: Crit pts $\sim$ cells of dim. index!

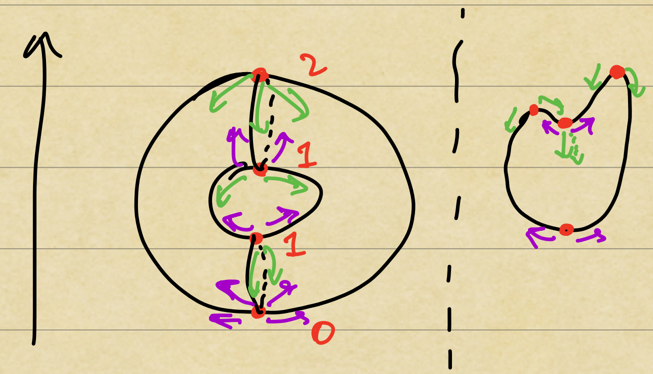
$$\mathcal{U}(p) := \{ q \mid \exists\, \gamma : (-\infty, 0] \to X \quad \gamma \in C^1$$
$$\text{s.t. } \gamma'(t) = -\nabla f(\gamma(t))$$
$$\lim_{t \to -\infty} \gamma(t) = p \text{ and } \gamma(0) = q \}$$

unstable mfld

$$S(p) = \{ q \mid \exists\, \gamma : [0,\infty) \to X \quad \gamma \in C^1$$
$$\text{s.t. } \gamma'(t) = -\nabla f(\gamma(t))$$

Stable mfld $\qquad \lim_{t\to+\infty} \gamma(t) = p \quad \& \quad \gamma(0) = q \}$

eg: height fn. on Torus f



Fact:
- $\mathcal{U}(p)$ & $S(p)$ are homeo. to open disks
- $\dim \mathcal{U}(p) = k$
- $\dim S(p) = n-k$ $\quad$ where $\quad n = \dim X$

Let $\quad \mathcal{U}(p) \cong D_1(0)$ so, we have $\mathcal{U}'(p) \subset \mathcal{U}(p)$ corresponding to $D_{1/2}(0)$

Hypothesis: $\partial \mathcal{U}'(p) \cap S(q)$ is a mfld of dim $\quad i(p) - i(q) - 1$.

Defn: Morse Chain complex

$$C_k(X,f) := \mathbb{Z}\langle c : \operatorname{ind}(c)=k \ \& \ c \in \operatorname{Crit}(f)\rangle$$
$$= \mathbb{Z}\langle e_i^k \rangle$$

$$d_k : C_k(X,f) \to C_{k-1}(X,f)$$

$$d_k e_i^k = \sum_j c_{ij}\, e_j^{k-1}$$

where c_{ij} = signed # of $\partial \mathcal{U}'(e_i^k) \cap S(e_j^{k-1})$

Claim: $d^2 = 0$ and $\ker d_k / \operatorname{Im} d_{k+1} \cong H_k(X;\mathbb{Z})$

Sketch of proof:

The key obs. is that f gives a CW cpx str. on X with the critical pt. e_i^k corresponding to a k-cell corresponding to $\mathcal{U}(e_i^k)$.

Then, d_k = cellular boundary map for this CW str. ⊠

Short-comings w/ infinite dim :

- Analytic difficulties : $\mathcal{U}(p)$ & $S(p)$ are disks
- Satisfying Hypothesis : Cumbersome to formulate and prove $\partial\mathcal{U}'(p) \cap S(q)$ .... for appropriate (g,f)
- Proving $d^2=0$: Preferable to have intrinsic proof.

Towards a Reformulation:

Let $u \in \mathcal{U}(p_-) \cap S(p_+)$

$\Rightarrow$ $\exists\ \gamma_- : (-\infty, 0] \to X$ & $\gamma_+ : [0,\infty) \to X$

s.t. $\gamma_-(0) = u = \gamma_+(0)$; $\gamma_\pm'(t) = -\nabla f(\gamma_\pm(t))$

$\lim_{t\to\pm\infty} \gamma_\pm(t) = p_\pm$

$\Rightarrow \; \exists \gamma : \mathbb{R} \to X$ s.t. $\lim_{t \to \pm\infty} \gamma(t) = p_\pm$ and $\gamma = \begin{cases} \gamma_- & \text{if } t<0 \\ \gamma_+ & \text{if } t>0 \end{cases}$

and $\gamma'(t) = -\nabla f(\gamma(t))$

$\mathcal{M}(p_-, p_+)$

Note that $(s \cdot \gamma)(t) = \gamma(t+s)$ gives a $\mathbb{R}$ action on $\mathcal{M}(p_-, p_+)$

$\mathbb{R}$ action is free iff $p_- \neq p_+$

Let $\check{\mathcal{M}}(p_-, p_+) := \mathcal{M}(p_-, p_+)/\mathbb{R}$

Obs: $(\partial \mathcal{U}'(p)) \cap S(q) \longrightarrow \check{\mathcal{M}}(p, q)$

when $i(p) = i(q) + 1$ is a bijection.

$$\Rightarrow \quad d e_i^k = \sum_j \# \check{\mathcal{M}}(e_i^k, e_j^{k-1}) \, e_j^{k-1}$$

Describing $\mathcal{M}$ as a "zero set":

Fix $p_\pm \in \mathrm{Crit}(f)$

Then, $P(p_-, p_+) := \{ \underset{\gamma \in C^1}{\gamma : \mathbb{R} \to X} \mid \lim_{t \to \pm\infty} \gamma(t) = p_\pm \}$

Now, let

$$\mathcal{E}_\gamma := \Gamma_0(\gamma^* TX) = T_\gamma P$$

i.e.,

$$\varphi \in \mathcal{E}_\gamma \iff \varphi(t) \in T_{\gamma(t)} X \quad \forall t \in \mathbb{R}$$

$$\& \quad \lim_{t \to \pm\infty} \varphi(t) = 0$$

$$\leadsto \quad \begin{array}{c} \mathcal{E} \\ \downarrow \pi \\ P(p_-, p_+) \end{array} \quad \text{is a vector bundle}$$

with $\pi^{-1}(\gamma) = \mathcal{E}_\gamma \subset \mathcal{E}$

$\leadsto \mathcal{F}(\gamma)(t) := \gamma'(t) + (\nabla f)(\gamma(t))$

is a section of $\mathcal{E}$!, i.e.,

$$\begin{tikzcd} & \mathcal{E} \arrow[dr, "\pi"] & \\ P \arrow[ur, "\mathcal{F}"] \arrow[rr, "="] & & P \end{tikzcd}$$

$$\mathcal{M}(p_-, p_+) = \mathcal{F}^{-1}(0)$$

hypothesis is satisfied $\iff$ $\mathcal{F}$ is transverse to 0!

Upshot: $\mathcal{M}(p_-, p_+)$ & hence $\check{\mathcal{M}}(p_-, p_+)$ is a manifold!

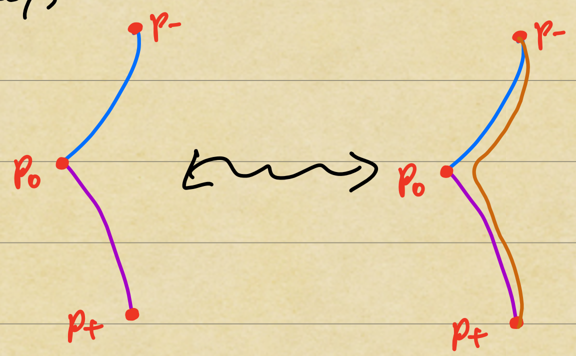
Compactness, Gluing & $d^2=0$:

Observe that

$$\langle d^2 e_i^k, e_j^{k-2}\rangle = \sum_r \#\check{\mathcal{M}}(e_i^k, e_r^{k-1})\cdot\#\check{\mathcal{M}}(e_r^{k-1}, e_j^{k-2})$$

Hence $d^2=0 \iff 0 = \sum_r \#\check{\mathcal{M}}(e_i^k, e_r^{k-1})\cdot\#\check{\mathcal{M}}(e_r^{k-1}, e_j^{k-2})$

Heuristically,



i.e., $\check{\mathcal{M}}(p_-,p_0)\times\check{\mathcal{M}}(p_0,p_+) \subset$ " $\underbrace{\check{\mathcal{M}}^+(p_-,p_+)}$ "

compactification of $\check{\mathcal{M}}(p_-,p_+)$

Fact: $\check{\mathcal{M}}(p_-,p_+)$ has a compactification $\check{\mathcal{M}}^+(p_-,p_+)$.

If $i(p_-)=i(p_+)+2$, $\partial\check{\mathcal{M}}^+(p_-,p_+) = \bigcup_{\substack{p_0 \\ i(p_-)<i(p_0)<i(p_+)}} \check{\mathcal{M}}(p_-,p_0)\times\check{\mathcal{M}}(p_0,p_+)$

$\Rightarrow \quad \check{\mathcal{M}}^{+}(e_i^k, e_j^{k-2})$ is a cpct 1-mfld

$\Rightarrow \quad 0 = \# \partial \check{\mathcal{M}}^{+}(e_i^k, e_j^{k-2}) = \sum_r \# \check{\mathcal{M}}(e_i^k, e_r^{k-1}) \cdot \# \check{\mathcal{M}}(e_r^{k-1}, e_j^{k-2})$

Example:

Let $X = \mathbb{C}P^n = \mathbb{C}^{n+1} \setminus \{0\} / \mathbb{C}^*$

$= S^{2n+1} / S^1$

The metric on S^{2n+1} descends to X, say g.

Fix $L : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ self adj. w/ distinct eigenvalues

Let $\Lambda(v) = \dfrac{\langle Lv, v \rangle}{\|v\|^2}$ ← real inner product

$\Rightarrow \frac{1}{2}\Lambda(v) : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{R}$ is inv. under $\mathbb{C}^*$

$\Rightarrow$ it descends to a fn.

$f : \mathbb{C}P^n \to \mathbb{R}$

Now, $(\nabla \Lambda|_{S^{2n+1}})(w) = Lw - \Lambda(w) w$

as L is $\mathbb{C}$-linear,

$w \in \mathrm{Crit}(f) \iff Lw = \Lambda(w) w$

i.e., w is an eigenvector

Let $\quad L w_i = \lambda_i w_i \qquad \lambda_1 < \lambda_2 \cdots\cdots \lambda_{n+1}$

$$\Rightarrow \quad \underset{w_i}{\mathrm{Hess}}(f) = L - \lambda_i \Big|_{(\mathbb{C}w)^\perp}$$

$$\Rightarrow \quad i(w_i) = 2(i-1)$$

Now, the gradient flow line

$$\gamma'(t) = -\nabla f(\gamma(t))$$

$$\Updownarrow$$

$$\gamma(t) = [z(t)] \qquad z : \mathbb{R} \to \mathbb{C}^{n+1} \setminus \{0\}$$

with $\quad z'(t) = -L z(t)$

$$\Rightarrow \quad z(t) = \sum_{i=1}^{n+1} \alpha_i e^{-\lambda_i t} w_i$$

$$\Rightarrow \quad \mathcal{M}(w_j, w_i) = \left\{ \begin{array}{l} \alpha_1 = \alpha_2 \cdots \alpha_{i-1} = 0 \quad \alpha_i \neq 0 \\ \alpha_{n+1} = \alpha_n \cdots \alpha_{j+1} = 0 \quad \alpha_j \neq 0 \end{array} \right\} \Big/ \mathbb{C}^*$$

$$\cong \mathbb{C}P^{j-i} \setminus \left(\mathbb{C}P^{j-i-1} \sqcup \mathbb{C}P^{j-i-1} \right)$$

$$\Rightarrow \quad \check{\mathcal{M}}(w_j, w_i) \cong \mathbb{C}^{j-i-1} \times S^1 \cong D_1(0)^{j-i-1} \times S^1$$

$$\& \quad \check{\mathcal{M}}^+(w_j, w_i) \cong \overline{D_1(0)}^{j-i-1} \times S^1$$