Chern-Simons Inv & Rep Spaces

Review of Gauge Theory: Fix a cpct mfld X w/ a hermitian bundle $E \downarrow X$ and a preferred triv. $\Lambda^2 E \cong \underline{\mathbb{C}}$

Ref: Taubes – Differential Geometry

$$\rightsquigarrow \mathrm{Aut}(E) = \{ g \in \Gamma(X, \mathrm{End}(E)) \mid g^* g = \mathrm{Id}_E \text{ and } \det(g) = 1 \}$$

will denote by $\mathcal{G}_E$

$\rightsquigarrow (\mathfrak{g}_E)_x :=$ traceless endo. of $E_x = \mathrm{End}_0(E_x)$

$\dashv \ \mathfrak{g}_E = \mathrm{End}_0(E)$

Fact: $T_{\mathrm{Id}} \mathcal{G}_E = \Gamma(X, \mathfrak{g}_E)$

Defn (connection): A connection A is a linear map $\nabla_A : \Gamma(X, E) \to \Gamma(X, E \otimes T^*X)$ s.t.

$$\nabla_A(fs) = df \otimes s + f \nabla_A s$$

We will restriction attention to connections A s.t.

$$d\langle s,t\rangle_E = \langle \nabla_A s, t\rangle_E + \langle s, \nabla_A t\rangle_E$$

& $d(s_1 \wedge s_2) = \nabla_A s_1 \wedge s_2 + s_1 \wedge \nabla_A s_2$

We call these connections $SU(2)$ connections and denote $\Gamma(X, E\otimes \Lambda^k T^*X)$
$= \Omega^k(X,E)$

Facts: • If A & A' are two $SU(2)$ connections, then,
$\exists\, a \in \Omega^1(\mathfrak{g}_E)$ s.t.

$$\nabla_{A'} s = \nabla_A s + as \qquad \forall s \in \Omega^0(X,E)$$

we say $A' = A + a$ in this case

• ∇_A extends to a map $d_A : \Omega^k(X,E) \to \Omega^{k+1}(X,E)$ $\forall k$ s.t.

$$d_A s = \nabla_A s \quad \text{when } s \in \Omega^0(X,E)$$

$$d_A(\alpha \wedge \beta) = (d_A \alpha)\wedge \beta + (-1)^k \alpha \wedge d\beta$$

$$\alpha \in \Omega^k(X,E) \qquad \beta \in \Omega^l(X)$$

• $\exists\, F_A \in \Omega^2(\mathfrak{g}_E)$ s.t.

$$d_A^2 s = F_A s \qquad \forall s \in \Omega^0(X,E)$$

• If $A' = A + a$, then

$$F_{A'} = F_A + d_A a + a \wedge a$$

Denote the space of SU(2) connections as $\mathcal{A}_E$

Then $\mathcal{G}_E \circlearrowright \mathcal{A}_E$ as follows:

$$\nabla_{g\cdot A}(gs) = g(\nabla_A s) \quad \text{, i.e., a left action!}$$

$$\Rightarrow \nabla_{g\cdot A} s = g(\nabla_A(g^{-1}s))$$
$$= g(\nabla_A(g^{-1})s + g^{-1}\nabla_A s)$$
$$= \nabla_A s + g(-g^{-1}(\nabla_A g)g^{-1}s)$$
$$= \nabla_A s - (\nabla_A g)g^{-1}s$$
$$\Rightarrow g\cdot A = A - (\nabla_A g)g^{-1}$$

Further, if $A' = g\cdot A$,

$$F_{A'}(gs) = d_{A'}(d_{A'}(gs))$$
$$= d_{A'}(g(d_A s))$$
$$= g(d_A^2 s)$$
$$= g(F_A s)$$
$$\Rightarrow F_{g\cdot A} = g F_A g^{-1}$$

Upshot: $F_A = 0 \Rightarrow F_{g\cdot A} = 0 \quad \forall g \in \mathcal{G}_E$

Chern-Simons Invar: Let Y be a 3-mfld, oriented, closed & $E \downarrow Y$ be the trivial $SU(2)$ bundle

$\Rightarrow \exists$ a 4-mfld X cpct, oriented s.t. $\partial X = Y$.

E extends on X, call it $\tilde{E}$.

If A is a conn on E, we can extnd it to $\tilde{E}$, say $\tilde{A}$.

Then $CS : \mathcal{A}_E \longrightarrow \mathbb{R}/\mathbb{Z}$ is defined by

$$CS(A) = \frac{1}{8\pi^2} \int_X tr(F_{\tilde{A}}^2) \quad \text{mod } \mathbb{Z}$$

This is well defined for the follo reason:

Let $(X', \tilde{E}', \tilde{A}')$ be another extension of (Y, E, A)

Then $Z = X \cup_Y X'$, $F = \tilde{E} \cup \tilde{E}'$, $B = \tilde{A} \cup \tilde{A}'$

is a closed mfld ....

& so,

$$\frac{1}{8\pi^2} \int_Z tr(F_B^2) = \langle c_2(F), [Z] \rangle \in \mathbb{Z}$$

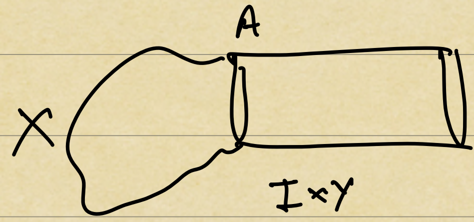
for $c_1(F) = 0$

(recall $c_1(\tilde{E}) = c_1(\tilde{E}') = 0$)

In fact,

$$CS : \mathcal{A}_E/\mathcal{G}_E \longrightarrow \mathbb{R}/\mathbb{Z}$$

for the following reason:



$g\cdot A \Rightarrow \exists B$ on $I\times Y$ s.t.
$B|_{\{0\}\times Y} = A$ & $B|_{\{1\}\times Y} = g\cdot A$

$$\Rightarrow \quad CS(g\cdot A) - CS(A) = \frac{1}{8\pi^2}\int_{I\times Y} tr(F_B^2)$$

$$= \frac{1}{8\pi^2}\int_{S^1\times Y} tr(F_B^2)$$

$$= deg(g) \qquad g : Y \longrightarrow SU(2)$$

$$= 0 \quad (\mathrm{mod}\ \mathbb{Z})$$

Denote $\mathcal{B}_E = \mathcal{A}_E/\mathcal{G}_E$ & let g be a metric on Y.

We first equip $\mathcal{A}_E$ with a metric:

$$T_A\mathcal{A}_E = \Omega^1(Y, E)$$

$$\langle \alpha, \beta\rangle = \int_Y -tr(\alpha\wedge *\beta)$$

then $* : \Omega^k \longrightarrow \Omega^{3-k}$

s.t. $\omega\wedge *\tau = \langle \omega, \tau\rangle_{\Lambda^k}\, dvol_g$

If $\{A_t\}$ is a path in $\mathcal{A}_E$, we get a path $\{B_t\}$ in $\mathcal{A}_{\tilde{E}}$

$$\Rightarrow \quad D_A CS(\alpha) = \frac{\partial}{\partial \varepsilon} \frac{1}{8\pi^2} \int_X tr(F^2_{B+\varepsilon b}) \Big|_{\varepsilon=0}$$

$$= \frac{1}{8\pi^2} \int_X tr(d_B b \wedge F_B + F_B \wedge d_B b)$$

Now, $d_B(b \wedge F_B) = d_B b \wedge F_B - b \wedge d_B F_B$
$= d_B b \wedge F_B$

$$\Rightarrow \int_X tr(F_B \wedge d_B b) = \int_X tr(d_B b \wedge F_B) = \int_X tr(d_B(b \wedge F_B))$$

$$= \int_X d(tr(b \wedge F_B))$$

$$= \int_Y tr(\alpha \wedge F_A)$$

$$\Rightarrow \quad D_A CS(a) = \frac{1}{4\pi^2} \int_Y tr(\alpha \wedge F_A)$$

$$= -\frac{1}{4\pi^2} \langle a, *F_A \rangle$$

$\Rightarrow \quad \operatorname{grad} CS(A) = -\frac{1}{4\pi^2} * F_A$

$\Rightarrow \operatorname{Crit}(CS) =$ Flat connections

$\Rightarrow \operatorname{Crit}(CS) \subset \mathcal{A}_E / \mathcal{G}_E$

is $R_{SU(2)}(Y) := \operatorname{Hom}(\pi_1 Y, SU(2)) / SU(2)$

Conj.: $R^*_{SU(2)}(Y) \neq 0$ $\forall$ Y is a $\mathbb{Z}HS^3$

Fact: • $R^*_{SL(2,\mathbb{C})}(Y) \neq 0$ $\forall$ Y is a $\mathbb{Z}HS^3$

[Zentner '18]

• $\exists$ Y a closed hyp. mfld & Y a $\mathbb{Q}HS^3$

s.t. $R^*_{SU(2)}(Y) = 0$

[Cornwell '15 $\Rightarrow$ $Y = \Sigma_2(8_{18})$ has this property

Bonahon, Siebenmann '10 $\Rightarrow$ Y is hyperbolic]

Examples:

- Lens spaces:

Thm (Fintushel-Stern '85, Kirk-Klassen '90): Let $Y = L(p,q)$

and $\rho_n : \pi_1 Y \longrightarrow SU(2)$

$$\langle \gamma \rangle \longrightarrow \begin{pmatrix} e^{2i\pi n/p} & 0 \\ 0 & e^{-2i\pi n/p} \end{pmatrix}$$ where γ is a generator.

$\Rightarrow$ $R_{SU(2)}(Y) = \{ \rho_n \mid n \in \mathbb{Z} \ \& \ 0 \leq n \leq \frac{p}{2} \}$

and

$$CS(\rho_n) = -\frac{n^2 r}{p} \pmod{\mathbb{Z}}$$

where $r \in \mathbb{Z}$ s.t. $qr \equiv -1 \pmod{p}$

Rmk: $L(p_1,q_1) \simeq L(p_2,q_2)$ (homotopy eq.)

$\Updownarrow$

$p_1 = p_2$ and $q_1 q_2 \equiv \pm k^2 \pmod{p_1}$ for some $k \in \mathbb{N}$

$\Updownarrow$

$\{CS(\rho)\}_{\rho \in R_{SU(2)}(L(p_i,q_i))}$ are same.

• Seifert Fibered space : (Fintushel, Stern '85; Furuta '90)

Let p, q, r be pair-wise co-prime

$$\Sigma(p,q,r) := S^5 \cap \{ z_1^p + z_2^q + z_3^r = 0 \} \subset \mathbb{C}^3$$

Then, $S^1 \curvearrowright \Sigma(p,q,r)$ by $u \cdot (z_1, z_2, z_3) = (u^{qr} z_1, u^{pr} z_2, u^{pq} z_3)$

$$\begin{tikzcd} S^1 \arrow[r] & \Sigma(p,q,r) \arrow[d] \\ & S^2(p,q,r) \end{tikzcd}$$

$S^2(p,q,r)$ ⇝ orbifold sphere with 3 singular pt. of order p,q,r

$$1 \to \pi_1 S^1 \to \pi_1 \Sigma(p,q,r) \to \pi_1^{orb} S^2(p,q,r) \to 1$$

$$T_{p,q,r} := \{ x, y, z \mid x^p = y^q = z^r = xyz = 1 \} = \pi_1(S^2(p,q,r))$$

$$\pi_1 S^1 \subset Z(\pi_1 \Sigma(p,q,r))$$

Facts:

- Any $\rho : \pi_1 \Sigma(p,q,r) \to SU(2)$, non-trivial, has $\rho(\pi_1 S^1) \subset Z(SU(2)) = \{\pm 1\}$
- $\exists\ \Psi : SU(2) \to SO(3)$ group homo. s.t. Ψ is surj. with kernel $Z(SU(2))$

- $R_{SU(2)}(\Sigma(p,q,r)) \xrightarrow{\psi_*} R_{SO(3)}(\Sigma(p,q,r))$ is a bijection (This follows from $H_1(\Sigma(p,q,r);\mathbb{Z}/2)=0$)

$\Rightarrow R_{SU(2)}(\Sigma(p,q,r)) \cong R_{SO(3)}(\Sigma(p,q,r))$
$\cong R_{SO(3)}(S^2(p,q,r))$
$= \{\bar\rho : T_{p,q,r} \to SO(3)\}$

Suppose $\bar\rho \in R_{SO(3)}(S^2(p,q,r))$ & $\bar\rho \neq 1$.

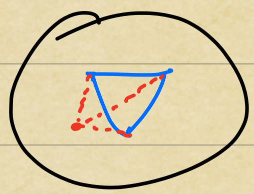
Let $X = \bar\rho(x)$, $Y = \bar\rho(y)$, $Z = \bar\rho(z)$

$\Rightarrow X$ is a rotation along an axis l_x of angle $\frac{2\pi k}{p}$

Y " l_y " $\frac{2\pi l}{q}$

Z " l_z " $\frac{2\pi m}{r}$

As $\bar\rho \neq 1$, l_x, l_y, l_z are non-coplanar

Now $XYZ = 1 \Leftrightarrow$ 

This constraint gives certain ineq. about k, l, m.

Upshot: $\# R_{SU(2)}(\Sigma(p,q,r)) = 1 - \frac{\sigma(B(p,q,r))}{4}$

where $B(p,q,r) = \overline{D^6} \cap \{z_1^p + z_2^q + z_3^r = \varepsilon\} \subset \mathbb{C}^3$

and σ denotes signature of $\cup : H^2(\ldots;\mathbb{Z}) \times H^2(\ldots;\mathbb{Z}) \to H^4(\ldots;\mathbb{Z}) \cong \mathbb{Z}$

Now, let A be a flat $SU(2)$ connection on $\Sigma(p,q,r)$

Consider the mapping cylinder M_φ of $\varphi : \Sigma(p,q,r) \to S^2(p,q,r)$

This is a 4-mfld containing 3-singular pts. These pts are cones on $L(p,p')$, $L(q,q')$, $L(r,r')$ where $p', q', r' \in \mathbb{Z}_{>0}$ are computable.

Let $W = M_\varphi \setminus (CL(p,p') \cup \cdots)$

Then, $\pi_1 W = T_{p,q,r}$

$\Rightarrow$ A extends to a flat $SO(3)$ connection on W

$$\Rightarrow \quad CS(A|_{\Sigma(p,q,r)}) \equiv CS(A|_{L(p,p')}) + CS(A|_{L(q,q')}) + CS(A|_{L(r,r')}) \pmod{\mathbb{Z}}$$

Thm (FS '85): Given any $\rho \in R_{SU(2)}(\Sigma(p,q,r))$, $\exists\, e \in \mathbb{Z}$ s.t.

$$CS(A_\rho) \equiv \frac{e^2}{4pqr} \pmod{\mathbb{Z}}.$$

Casson-Invar:

Recall that $\mathcal{B}_E = \mathcal{A}_E / \mathcal{G}_E$

If $A \in \mathcal{A}_E$,

$$\mathrm{Stab}(A) := \{ g \in \mathcal{G}_E \mid g \cdot A = A \} \supset \{ g \in \mathcal{G}_E \mid g(x) = \pm 1 \ \forall x \in Y \}$$
$$=: Z(\mathcal{G}_E)$$

We call these irred connections. Denote the subset $\mathcal{A}_E^*$

$\Rightarrow \mathcal{B}_E^* := \mathcal{A}_E^* / \mathcal{G}_E \subset \mathcal{B}_E$

Note that $\mathcal{G}_E / Z(\mathcal{G}_E)$ acts freely on $\mathcal{A}_E^*$

$\Rightarrow \mathcal{B}_E^*$ is a "manifold".

We want to restrict CS to $\mathcal{B}_E^*$

$\Rightarrow \mathrm{Crit}(CS|_{\mathcal{B}_E^*}) =: R^*_{SU(2)}(Y)$

$\Rightarrow R^*_{SU(2)}(Y) = R_{SU(2)}(Y) \cap \mathcal{B}_E^*$

We want to say $\# R^*_{SU(2)}(Y)$ is an "euler char."

This indeed makes sense in some cases.

Let $R^{red}_{SU(2)}(Y) = R_{SU(2)}(Y) \setminus R^{*}_{SU(2)}(Y)$

Then, the trivial connection $[\theta] \in R^{red}_{SU(2)}(Y)$

To make sure $R^{*}_{SU(2)}(Y)$ is a finite set of pts, we need to perturb CS. When this happens, some irred. crit pts could become reducible

To avoid this, we impose the following:

Hypothesis: $R^{red}_{SU(2)}(Y) = \{[\theta]\}$ Y connected

$$\Updownarrow$$

$$H_*(Y;\mathbb{Z}) = H_*(S^3;\mathbb{Z})$$

Proof: If $[\rho] \in R^{red}_{SU(2)}(Y)$, then $\exists$ repre. $\rho : \pi_1 Y \to U(1) \subset SU(2)$

Now, $0 = \mathrm{Hom}(\pi_1 Y, U(1)) = \mathrm{Hom}(\pi_1 Y, \mathbb{R}/\mathbb{Z})$

$= \mathrm{Hom}(H_1(Y;\mathbb{Z}), \mathbb{R}/\mathbb{Z})$

$\Rightarrow H_1(Y;\mathbb{Z}) = 0 \Rightarrow H^2(Y;\mathbb{Z}) = 0$ and $H^1(Y;\mathbb{Z}) = 0$

$\Rightarrow H_2(Y;\mathbb{Z}) = 0$ $\boxtimes$

We call such Y a $\mathbb{Z}HS^3$

Thm (Taubes '90): Let Y be a $\mathbb{Z}HS^3$.

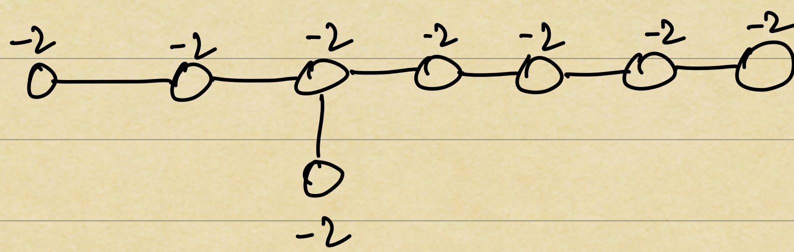
Then, after perturbing CS generically, we can define $\# R^*_{SU(2)}(Y)$ which doesn't depend on the perturbation. Further,

$$2\lambda(Y) = \# R^*_{SU(2)}(Y)$$

where $\lambda : \mathbb{Z}HS^3 \to \mathbb{Z}$ is the Casson Invariant.

eg:

- $\lambda(\Sigma(p,q,r)) = \frac{1}{8}\sigma(B(p,q,r))$ [F. S.]
- $B(2,3,5)$ has the follo intersection form



$\Rightarrow$ $\sigma(B(2,3,5)) = -8$

i.e., $\lambda(\Sigma(2,3,5)) = -1$

- **Seifert Fibred Spaces:**

Thm (FS '85): Let $Y = \Sigma(a_1, a_2, \ldots a_n)$ be a seifert fibred integer homology sphere.

Then, $\lambda(Y) = \frac{1}{2}\chi(R^*_{SU(2)}(Y))$

Rmk:
- $H_i(R^*_{SU(2)}(Y); \mathbb{Z})$ is free and 0 in odd i. [Kirk-Klassen '91]
- $R^*_{SU(2)}(Y)$ is a union of rational varieties [Bauer-Okonek '90]

- **Links of Singularities:**

Casson Invariant Conj: Let $(X, 0)$ be an isolated surface-link singularity with link Σ, a $\mathbb{Z}HS^3$ and Milnor fibre F.

Then $\lambda(\Sigma) = 1/8\,\sigma(F)$

[Neumann-Wahl '90]: wh Σ is a graph mfld, weighted homogeneous sing.

[Collin-Saveliev '01]: branched cover constructions

[Némethi-Okuma '05]: splice-type singularities

Flat Connections & Representations:

Let $X = I$ & $E \downarrow X$ trivial with some conn A.

Now, E has a trivial conn A_0.

Let $A = A_0 + a$

Then, A is flat as $F_A \in \Omega^2(X, E)$ & $\Lambda^2 X = 0$

Try & find $s : I \to E$

s.t. $\nabla_A s = 0$

$$s'(t) + a(t)s(t) = 0$$

a soln exists with any $s(0)$ & is uniquely determined by it.

$\Rightarrow \exists\, L : \mathbb{C}^n \to \mathbb{C}^n$ s.t.

$$L(s(1)) = s(0)$$

Fact: $L \in SU(n)$

Now, let X be any cpct mfld & fix $x_0 \in X$

Then, if

$$\gamma : [0,1] \to X \qquad \gamma(0) = \gamma(1) = x_0,$$

$\gamma^* E \downarrow [0,1]$ is a triv. bundle & so, we get a L_γ

as above. $\gamma = \gamma_1 \cdot \gamma_2 \Rightarrow L_\gamma = L_{\gamma_1} \cdot L_{\gamma_2}$

$$\Rightarrow \mathrm{Hol}_A : \Omega_{x_0} X \longrightarrow \mathrm{Aut}(E_{x_0}) \cong SU(n)$$

Fact: • $\langle w, (D_\gamma \mathrm{Hol}_A)(\xi)(v)\rangle = \int_0^1 \langle \tilde{w}, (F_A \tilde{v})(\dot{\gamma}(t) \wedge \xi(t))\rangle_E \, dt$

when $\tilde{v}, \tilde{w} : [0,1] \to \gamma^* E$ s.t. $\nabla_A \tilde{v} = \nabla_A \tilde{w} = 0$

• $F_A = 0$ on X

$\Rightarrow D_\gamma \mathrm{Hol}_A : T_\gamma \Omega_{x_0} X \longrightarrow \mathfrak{g}_{x_0}$

vanishes everywhere

Hence, when $F_A = 0$, we get a map

$\mathrm{Hol}_A : \pi_0(\Omega_{x_0} X) \to \mathrm{Aut}(E_{x_0})$

i.e., $\mathrm{Hol}_A : \pi_1(X, x_0) \longrightarrow SU(n)$