Floer Homology

Recap:

Want to mimic the following setup:

- $p \in \mathrm{Crit}(f) \Rightarrow \mathrm{Hess}_p f$ is non-degen.
- $C_k = \mathbb{Z}\langle c \in \mathrm{Crit}(f) \mid i(c) = k \rangle$

  and $d_k : C_k \to C_{k-1}$

  $\langle d_k p_1, p_2 \rangle = \# \check{\mathcal{M}}(p_1, p_2)$

  where we assume

  $\check{\mathcal{M}}(p_-, p_+)$ is a C^1-mfld of dim $i(p_-) - i(p_+) - 1$

  whenever $p_+ \neq p_-$ and admits a compactification

  $\check{\mathcal{M}}^+(p_-, p_+)$.
- Recall that $\check{\mathcal{M}}(p_-, p_+) = \mathcal{M}(p_-, p_+)/\mathbb{R}$

  &

  $\mathcal{M}(p_-, p_+) = \mathcal{F}^{-1}(0)$

  when

  $\mathcal{F} : \mathcal{P}(p_-, p_+) \to \mathcal{E}$

  $\gamma \longmapsto (t \mapsto \gamma'(t) + (\nabla f)(\gamma(t)))$

All connections will be irred unless stated otherwise

Non-degeneracy of Hess:

Recall that $\operatorname{grad} CS(A) = -*F_A$

$\Rightarrow \quad \mathrm{Hess}_A CS(a) = \partial_t(-*F_{A+ta})\big|_{t=0}$

$$= -*d_A a$$

But, $A \in \mathrm{Crit}(CS)$, then if $a = d_A \xi$,

$$\mathrm{Hess}_A CS(a) = -*d_A^2 \xi$$
$$= 0 \qquad (\because d_A^2 = F_A \wedge = 0)$$

In fact, $\ker d_A \subset \ker(\mathrm{Hess}_A CS)$

and $\mathrm{Im}\, d_A \subset \ker d_A$ when A is flat

Now, $T_{Id}\mathcal{G}_E = \Omega^0(Y, \mathfrak{g}_E)$

and $\mathcal{G}_E \to \mathrm{Aut}(\mathcal{A}_E)$ when linearised at Id,

$$T_{Id}\mathcal{G}_E \longrightarrow \mathrm{aut}(\mathcal{A}_E)$$
$$\xi \longmapsto -d_A\xi \qquad \text{here } e^{t\xi}\cdot A = A - e^{-t\xi} d_A e^{t\xi}$$

& $\mathrm{Im}\, d_A \subset \ker \mathrm{Hess}_A CS$ is merely reflecting that CS is $\mathcal{G}_E$-invariant!

Hypothesis (non-degen): $\operatorname{Im} d_A = \ker d_A$

Note: As $d_A^2 = 0$, we have a complex

$$\Omega^0(Y, \mathfrak{g}_E) \xrightarrow{d_A} \Omega^1(Y, \mathfrak{g}_E) \xrightarrow{d_A} \Omega^2(Y, \mathfrak{g}_E) \xrightarrow{d_A} \Omega^3(Y, \mathfrak{g}_E)$$

$\Rightarrow$ we have cohom groups $H^i_A = \ker d_A / \operatorname{Im} d_A$

Now, hypothesis $\Leftrightarrow$ $H^1_A = 0$

& irred of $A \Leftrightarrow T_{Id} \operatorname{Stab}(A) = H^0_A = 0$

More generally, we have the Hodge decomposition

Let $\Delta_A = d_A^* d_A + d_A d_A^* : \Omega^k \longrightarrow \Omega^k$

$$\Rightarrow \quad \Omega^k = (\ker \Delta_A) \oplus \operatorname{Im} d_A \oplus \operatorname{Im} d_A^*$$

is a L^2-orthogonal decomposition.

Under hypothesis,

$$\Omega^1 = \operatorname{Im} d_A \oplus \operatorname{Im} d_A^* = T_{Id} \mathcal{G}_E \oplus \ker d_A^*$$

$$(\ker d_A^* = \operatorname{Im} d_A^* \text{ as } \ker \Delta_A = 0)$$

$$\Rightarrow \underbrace{\{ A \in \mathcal{A}_E \mid \|A - A_0\| < \varepsilon \ \& \ d_{A_0}^*(A - A_0) \}}_{\text{Coulomb-Gauge}} \longrightarrow \mathcal{A}_E / \mathcal{G}_E \quad \text{is a chart!}$$

Further, $\mathrm{Hess}_A CS$ is self-adj., hence, we ought to consider $\mathrm{Hess}_A CS : \ker d_A^* \to \ker d_A^*$

Defn $D_A : \Omega^0(Y, \mathfrak{g}_E) \oplus \Omega^1(Y, \mathfrak{g}_E) \to \Omega^0(Y, \mathfrak{g}_E) \oplus \Omega^1(Y, \mathfrak{g}_E)$

$$D_A = \begin{pmatrix} 0 & -d_A^* \\ -d_A & *d_A \end{pmatrix}$$

Then, $\ker(D_A) = H_A^0 \oplus H_A^1$ and D_A self adj.

Further, $\mathrm{Hess}_A CS : \ker d_A^* \to \ker d_A^*$ non-degen $\Leftrightarrow$ D_A an iso.

Defn: A is an acyclic flat connection iff $\ker(D_A) = 0$

Rmk: One can show that $\mathrm{Im}(D_A)$ is closed &

$$\begin{aligned} \mathrm{coker}(D_A) &= \Omega^0 \oplus \Omega^1 / \mathrm{Im}(D_A) \\ &\cong \ker(D_A^*) \quad \Rightarrow \text{if } A \text{ is non-degen} \\ &= \ker(D_A) \quad \text{then } D_A \text{ is an iso.} \end{aligned}$$

Assumption:
- We assume $H_*(Y; \mathbb{Z}) \cong H_*(S^3; \mathbb{Z})$ from now on.
- All connection, unless stated otherwise are irred.

Moduli of Traj.:

As $Hess_A(CS) = -*d_A$, one can show that spectrum is unbounded in either directions.

So "$i(A)$" never makes sense!

Never the less, we try to construct F and analyse it.

Let $\{A_t\}$ be a path of connections, i.e., $\gamma(t) = A_t$

$$\gamma'(t) + \operatorname{grad} CS(\gamma(t)) = 0$$

$$\Updownarrow$$

$$\dot{A}(t) = *_3 F_{A(t)}$$

If $B = \partial_t + A(t)$ on $\mathbb{R}\times Y$

then, $F_B = -\dot{A}(t)\wedge dt + F_{A(t)}$

$\Leftrightarrow \quad *_4 F_B = -F_B$

$\Leftrightarrow \quad F_B^+ = 0$

Now, F_B is invariant under $\mathbb{R}\times Y$ gauge trans!

To exploit this, we change our path space to the follo:

$$\mathcal{P}(A_-, A_+) = \{ B \text{ conn. on } \mathbb{R}\times Y \mid B = \partial_t + A(t) + c\,dt$$

where $A(t) \to A_\pm$ as $t \to \pm\infty$

& $c(t) \to 0$ as $t \to \pm\infty \}$

$/\text{gauge}$.

and $\mathcal{E}_B := \Omega^+(\mathbb{R}\times Y, \mathfrak{g}_E) \oplus \Omega^0(\mathbb{R}\times Y, \mathfrak{g}_E)$

so the

Coulomb gauge fixing

$$\mathcal{F}(B) = F_B^+ \oplus d_{B_0}^*(B - B_0)$$

for some fixed $[B_0] \in \mathcal{P}(A_-, A_+)$

B in Coulomb gauge on $\mathbb{R}\times Y$

$$\Updownarrow$$

$$0 = \dot{c}(t) - d_{A_0}^*(A - A_0) \qquad A_0 \text{ is time independent}$$

$\Rightarrow$ If $B_0 = \partial_t + A_0(t)$,

$$\nabla_{B_0}\mathcal{F}\begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} \dot{b} \\ \dot{c} \end{pmatrix} + D_{B_0}\begin{pmatrix} b \\ c \end{pmatrix}$$

$$\Rightarrow \nabla_{B_0}\mathcal{F} = (\partial_t + D_{B_0})$$

$\Rightarrow$ $\operatorname{vdim} \mathcal{M}_z(A_-, A_+) = \operatorname{Ind}(\partial_t + D_{B_0})$

$=: gr_z(A_-, A_+)$

$z \in \pi_0(\mathcal{P}(A_-, A_+))$

defined by $z(t) = [A(t)]$

Facts:
- $gr_z(A_-, A_+)$ can be defined even when $A_\pm$ are not crit. pts. or even if they are not irred.
- $gr_{z_-}(A_-, A_0) + gr_{z_+}(A_0, A_+) = gr_{z_- * z_+}(A_-, A_+)$
- $gr_z(A, A) \in 8\mathbb{Z}$ when z is a closed path now!
- There is a unique $[\Theta] \in \mathcal{B}_E$, i.e., the triv. conn.

$\Rightarrow$ $i(A) \in \mathbb{Z}/8$ with

$i(A) \equiv gr_z(A, \Theta) \pmod 8$

makes sense when $A \in \operatorname{Crit}(CS) \cap \mathcal{B}_E^*$

$\Rightarrow$ $C_k(Y) := \mathbb{Z}\langle A \mid i(A) \equiv k \pmod 8,\ A \text{ flat irred.}\rangle$

& $d_k A = \sum_{\substack{(z,B) \\ gr_z(A,B)=1}} (\# \check{\mathcal{M}}_z(A,B)) B$

Why is this sum finite?