Construction & Properties of Instanton Floer Homology

Recap:

- **Hypothesis:** $CS : \mathcal{B}_E^* \to \mathbb{R}/\mathbb{Z}$ is Morse, i.e., $\forall A \in \mathrm{Crit}(CS)$ then we want $H^0_A = H^1_A = 0$

- Let θ denote the trivial connection, then,
$$C_k(Y) := \mathbb{Z}\langle A \mid A \text{ flat } \& \ k \equiv gr_z(\theta, A) \bmod 8 \rangle$$

- $$\langle d_k A_-, A_+ \rangle = \# \bigcup_{\substack{z \\ gr_z(A_-,A_+)=1}} \check{\mathcal{M}}_z(A_-, A_+)$$

Also, $z \in \pi_0(P(A_-, A_+)) \cong \pi_1(\mathcal{B}_E; A_-, A_+)$

& $\pi_1(\mathcal{B}_E; A_-, A_+) \cong \pi_1(\mathcal{B}_E; A) \cong \mathbb{Z}$

Q:
- Are there finitely many z with $\check{\mathcal{M}}_z(A_-, A_+) \neq \emptyset$?
- for z s.t. $\check{\mathcal{M}}_z(A_-, A_+) \neq \emptyset$, is $|\check{\mathcal{M}}_z(A_-, A_+)| < \infty$?

Assumption: All $\check{\mathcal{M}}_z(A_-, A_+)$ are transversely cut out, i.e.,
$$\nabla \mathcal{F}\big|_{\mathcal{M}_z(A_-, A_+)} : TP \to \mathcal{E}$$ is surjective everywhere.

Spectral Flow & Novikov Rings:

Define

$$SF : \pi_1(\mathcal{B}_E, A_0) \longrightarrow \mathbb{Z}$$

$$[z] \mapsto gr_z(A_0, A_0)$$

when A_0 irred.

In fact, by Atiyah-Singer-Patodi index theory, we can show

$$SF(z) = \frac{1}{\pi^2} \int_{\mathbb{R}\times Y} tr(F_B \wedge F_B) \quad \text{when } B = \partial_t + A(t)$$

when A_0 is irred flat and $A(t) = A_0$ if $|t| \gg 1$.

$\Rightarrow \pi_1(CS) : \pi_1(\mathcal{B}_E) \to \pi_1(\mathbb{R}/\mathbb{Z}) \cong \mathbb{Z}$

is proportional to SF

i.e., $SF = 8\pi_1(CS)$

Fact: $\pi_1(CS)$ is an isomorphism!

Upshot: If $\{z_i\}$ is an infinite seq in $\pi_1(\mathcal{B}_E; A_-, A_+)$

Then $|\pi_1(CS)(\tilde{z}_i)| \to \infty$

$\Rightarrow |SF(\tilde{z}_i)| \to \infty$

$\Rightarrow |gr_{z_i}(A_-, A_+)| \to \infty \Rightarrow$ only finitely many z_i contribute in d.

Compactness and Gluing:

We show both that $|\check{\mathcal{M}}_z(A_-,A_+)| < \infty$ if $\dim \check{\mathcal{M}}_z(A_-,A_+) = 0$ and $d^2 = 0$

Key result: (Uhlenbeck, Taubes)

If $K(A) := \frac{1}{8\pi^2}\int_X \mathrm{tr}(F_A^2) \leq 1$

$\forall [A] \in \check{\mathcal{M}}_z(A_-,A_+)$ (we assume $A_+ \neq A_-$ if $K(A)=1$)

Then $\check{\mathcal{M}}_z(A_-,A_+)$ can be compactified to a space $\check{\mathcal{M}}^+(A_-,A_+)$ with codim 1 strata $\bigcup_{A_0} \check{\mathcal{M}}_{z_-}(A_-,A_0) \times \check{\mathcal{M}}_{z_+}(A_0,A_+)$

where A_0 is potentially trivial

& $gr_{z_-}(A_-,A_0) + gr_{z_+}(A_0,A_+) = gr_z(A_-,A_+) + \dim H^0_{A_0}$

assuming A_- & A_+ are irred.

Claim: If A_- & A_+ are irred. with $gr_z(A_-,A_+) < 3$, $A_0 \neq \Theta$.

Proof: If $A_0 = \Theta$, $\dim \mathcal{H}^0_{A_0} = 3$ $\blacksquare$

Claim: If $A_- = A_0$ then $Z_- = [0]$ in $\pi_1(\mathcal{B}_E, A_-)$
and $[A(t)] \in \mathcal{M}_Z(A_-, A_0)$ is the constant path.

Proof: $$gr_{Z_-}(A_0, A_0) = \frac{1}{\pi^2}\int_{\mathbb{R}\times Y} tr(F_A \wedge F_A) = \frac{1}{\pi^2}\int_{\mathbb{R}\times Y} |F_A|^2 \, dvol.$$

$\Rightarrow$ $gr_{Z_-}(A_0, A_0) = 8k$ for some $k \in \mathbb{Z}$

$\Rightarrow$ $k = 0$

$\Rightarrow$ $F_A = 0$

$\Rightarrow$ $A(t) = A_0 \;\; \forall t$

$\Rightarrow$ $Z = \{0\}$ $\blacksquare$

Hence, $A_0 \neq A_+$ or A_- $\Rightarrow$ $gr_{Z_-}(A_-, A_0), gr_{Z_+}(A_0, A_+) > 0$

$$\Rightarrow \partial \check{\mathcal{M}}^+_Z(A_-, A_+) = \bigcup_{\substack{A_0 \\ gr_{Z_-}(A_-, A_0) = 1 \\ gr_{Z_+}(A_0, A_+) = 1}} \check{\mathcal{M}}_{Z_-}(A_-, A_0) \times \check{\mathcal{M}}_{Z_+}(A_0, A_+)$$

i.e., $d^2 = 0$.

Seifert Homology Spheres:

Thm (Fintushel-Stern '85): Let $Y = \Sigma(a_1, a_2, \ldots a_n)$ be a integer homology sphere.

Then $I_*(Y)$ is a free module over $\mathbb{Z}$ concentrated in even grading. In fact, if $R^*_{SU(2)}(Y) = \bigsqcup_\alpha R_\alpha$ is a a decomposition of the irred. rep. variety into connected components, then

$$I_*(Y) = \bigoplus_\alpha H_{*+\mu(\alpha)}(R_\alpha)$$

where $\mu(\alpha) \in 2\mathbb{Z}$.

examples:
- $I_*(\Sigma(2,3,6k+1)) = \mathbb{Z}_{(0)}^{\lfloor \frac{k}{2} \rfloor} \oplus \mathbb{Z}_{(2)}^{\lceil \frac{k}{2} \rceil} \oplus \mathbb{Z}_{(4)}^{\lfloor \frac{k}{2} \rfloor} \oplus \mathbb{Z}_{(6)}^{\lceil \frac{k}{2} \rceil}$
- $I_*(\Sigma(2,3,6k-1)) = \mathbb{Z}_{(0)}^{\lceil \frac{k}{2} \rceil} \oplus \mathbb{Z}_{(2)}^{\lfloor \frac{k}{2} \rfloor} \oplus \mathbb{Z}_{(4)}^{\lceil \frac{k}{2} \rceil} \oplus \mathbb{Z}_{(6)}^{\lfloor \frac{k}{2} \rfloor}$

Topological Constructions:

Admissible Bundles:

So far, we have talked about $SU(2)$ bundles on $\mathbb{Z}HS^3$.
Now, any $SO(3)$ bundle on a $\mathbb{Z}HS^3$ uniquely lifts to a $SU(2)$.

However, we can build most of the many $SO(3)$ bundles as follows:

Let Y be a cpct, conn., oriented 3-mfd & $E \to Y$ a $SO(3)$ bundle.

Then, we can get a bundle w/ fibre $SU(2)$:

we have $Ad : SU(2) \to Aut(SU(2))$

$$g \mapsto (h \mapsto ghg^{-1})$$

$\Rightarrow$ $Z(SU(2)) \subset \ker(Ad)$ and $SU(2)/Z(SU(2)) = SO(3)$

$\Rightarrow$ $Ad : SO(3) \longrightarrow Aut(SU(2))$

$\Rightarrow AdE := F(E) \times_{Ad} SU(2)$

is a bundle with fibres the lie group $SU(2)$

$\mathcal{G}_E := \Gamma(Y; \mathrm{Ad}\,E)$ acts on

$\mathcal{A}_E :=$ $SO(3)$ conn. on $E \downarrow Y$

Now, suppose Σ is a closed oriented surface $\subset Y$

w/ $\langle w_2(E), [\Sigma] \rangle \neq 0$ in $\mathbb{F}_2 := \mathbb{Z}/2$

Then, if $A \in \mathcal{A}_E$ is flat, $\mathrm{Stab}(A) = Z(\mathcal{G}_E)$:

If not, the holonomy rep $\rho_A : \pi_1 Y \to SO(3)$

has a rep. s.t. $\rho_A(\pi_1 Y) \subset O(2)$

Now, $w_1(E|_\Sigma) = w_1(E)|_\Sigma = 0$

$\Rightarrow \rho_{A|_\Sigma} : \pi_1 \Sigma \to SO(2) \cong U(1)$

$\Rightarrow (E|_\Sigma, A|_\Sigma)$ is a flat cplx line bundle

$\Rightarrow c_1(E|_\Sigma) = 0$

But,

$\langle c_1(E|_\Sigma), [\Sigma] \rangle \pmod 2 = \langle w_2(E|_\Sigma), [\Sigma] \rangle = \langle w_2(E), [\Sigma] \rangle \neq 0$

$\Rightarrow H^0_A = 0 \quad \forall A \in \mathrm{Crit}(CS)$

To ensure $H^1_A = 0$ $\forall A \in \mathrm{Crit}(CS)$, one needs to perturb!

Upshot: Can define $I^w_*(Y)$ when $[w] =$ P.D.$(w_2(E))$ is a closed 1-mfd.

Connect Sums:

One may not be able to find $\Sigma \subset Y$ with the prop. above. In this case, we can artificially force it:

$$Y' := Y \# T^3$$

$$E' := \underline{\mathbb{R}}^3 \# E_\delta$$

where $E_\delta = \underline{\mathbb{R}} \oplus L$ where L is a cplx line bundle with $c_1(L) = P.D.[\delta]$ where $\delta \subset T^3$ is a co-ordinate circle.

If Σ = section spanned by other circles

then $\langle w_2(E_\delta), [\Sigma] \rangle = \langle c_1(L), [\Sigma] \rangle \mod 2$

$= \# \Sigma \cap \delta \mod 2$

$= 1 \mod 2$

Upshot: $I^{\#}_{*}(Y) := I^{\delta}_{*}(Y')$ where $* = 0, 1, 2, 3$ is well defined.

Obs: Let $R^{\#}(Y) = \mathrm{Hom}(\pi_1 Y, SU(2))$