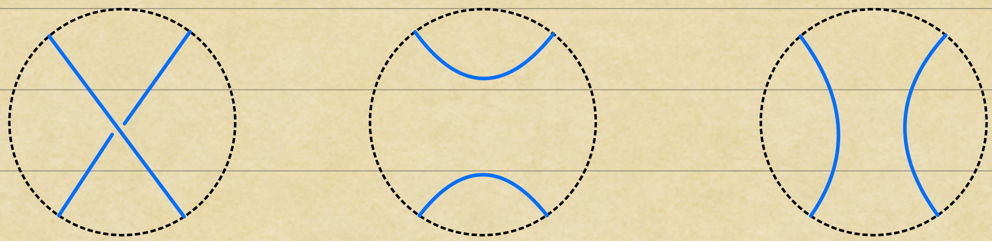
Exact Triangles in Instanton Theory

Skein Exact Triangle:



L_2 L_0 L_1

"unoriented skein relation"

Thm (Kronheimer–Mrowka '10):

$$\begin{tikzcd} I^{\#}(S^3, L_1) \arrow[r] & I^{\#}(S^3, L_0) \arrow[dl] \\ I^{\#}(S^3, L_2) \arrow[u] & \end{tikzcd}$$

where all maps are defined by cobordisms.

Defn: (Dehn surgery) Let $K \subset Y$ be a framed knot, i.e., we have a smth embedding

$$i: S^1_\lambda \times D^2_\mu \longrightarrow Y$$

s.t. $i(S^1 \times \{0\}) = K$

$$Y_{p/q}(K) = \left(Y \setminus i(S^1 \times D^2)\right) \cup_{i \circ f} (S^1 \times D^2)$$

s.t. $f: S^1_\theta \times \partial D^2_\tau \longrightarrow S^1_\lambda \times \partial D^2_\mu$

is a diffeo. with

$$f_*([\partial D^2_\tau]) = p[\partial D^2_\mu] + q[S^1_\lambda]$$

Thm (Floer): Let $K \subset Y$ be a framed knot and let

$$\mu = i(\{*\} \times \partial D^2).$$

Further, let $\lambda \subset Y \setminus K$ be a compact 1-mfld

Then, the follo exact triangle holds assuming all pairs are admissible

$$\begin{tikzcd}
I(Y_0(K); \lambda \cup \mu) \arrow[rr] & & I(Y_1(K); \lambda) \arrow[dl] \\
& I(Y; \lambda) \arrow[ul] &
\end{tikzcd}$$

Moreover, each map above is defined by a cobordism.

eg: Let $K \subset S^3$ be the Right handed Trefoil with Seifert framing

$\Rightarrow$ $S^3_1(K)$ = Poincaré hom sphere

$H_*(S^3_0(K)) = H_*(S^2 \times S^1)$ & $H_1(S^3_0(K)) = \mathbb{Z}\langle \mu \rangle$

& so,

$$I_*(S^3_0(K); \mu) = \mathbb{Z}^2_{(0)}$$

$\Rightarrow$ $S^3_0(K) \not\cong S^2 \times S^1$

Proof Strategy:

Use "neck-stretching" arguments with the follo lemma:

Lemma (Triangle detection lemma [Seidel, Osz-Sza..]):

Let $f_i : C_i \to C_{i+1}$ be chain maps where $i \in \mathbb{Z}/3$ & (C_i, d_i) are chain complexes.

Further, suppose we have chain homotopies $h_i : C_i \to C_{i+2}$ with

$$f_{i+1} \circ f_i = \partial h_i + h_i \partial$$

and that

$$f_{i+2} h_i - h_{i+1} f_i : C_i \to C_i$$

is a quasi-iso.

Then, $f_i \oplus h_i : C_i \to \mathrm{Cone}(f_{i+1})$ is a quasi-iso. $\blacksquare$

Neck-Stretching:

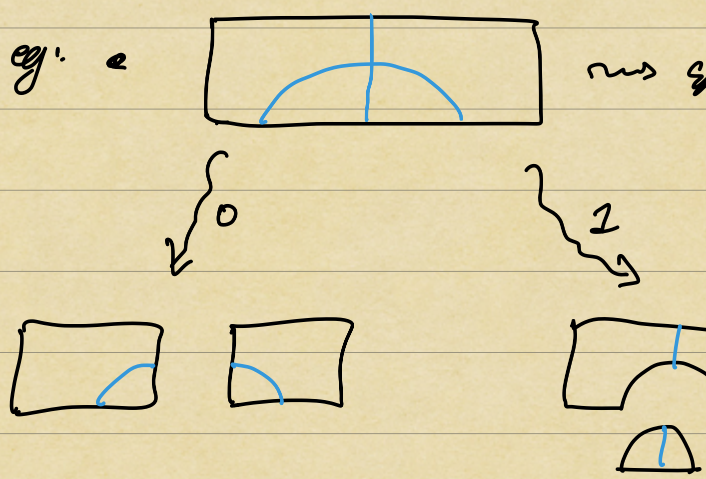
Let W be a 4-mfld with $\partial W = -Y_0 \sqcup Y_1$ and let $\mathcal{H}$ be a collection of closed connected hypersurfaces in the interior of W.

Then, we can form a family of metrics $G(\mathcal{H})$ with the follo properties:

(i) $G(\mathcal{H})$ is a stratified space

(ii) codim 1 strata correspond to "breaking" along a hypersurface

eg: •



$\rightsquigarrow$ space of metrics $= [0,1]$

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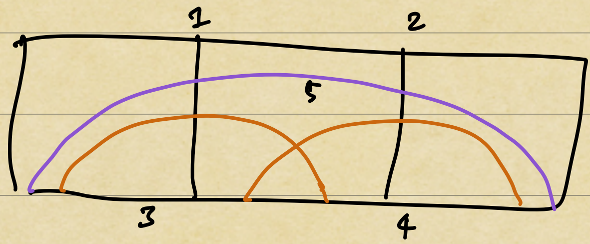
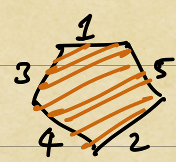


$\rightsquigarrow G$



Given such a space of metrics $G(H)$,
we can count solutions of formal dim $-\dim G(H)$
to define

$$m_{G(H)} : C_*(Y_0) \longrightarrow C_*(Y_1)$$

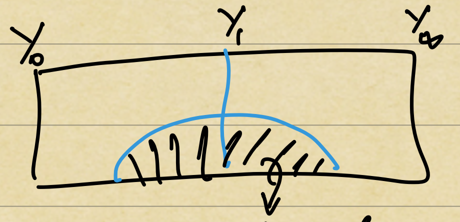
Counting $-\dim G(H)+1$ solutions over $G(H)$, one can deduce,

$$\partial m_{G(H)} + m_{G(H)}\partial = m_{\partial G(H)}$$

sliding off ends   breaking along a hypersurface

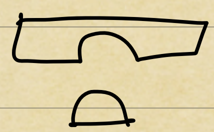
Upshot:

We will apply the alg. lemma to the map defined
by the cobordism

h_0: 

$\rightsquigarrow G(H) \cong [0,1]$

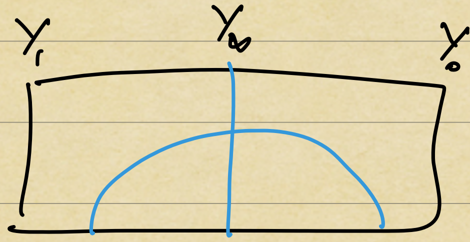
union of core & cocore handles

$\Rightarrow \partial h_0 + h_0 \partial = f_1 f_0 +$ map defined by 

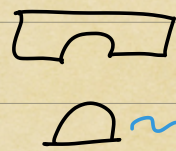
$\overline{\mathbb{CP}^2} \setminus D^4$

$= 0$

Similarly,

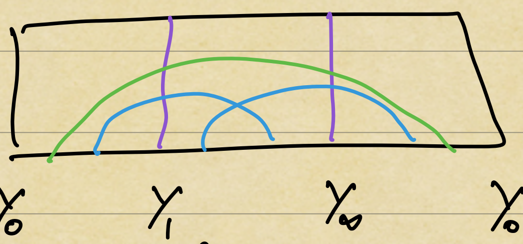
h_1 : 

$\rightsquigarrow G(H) \cong [0,1]$

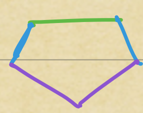
$\to$ $\partial h_1 + h_1 \partial = f_2 \circ f_1 +$ map defined by  $\rightsquigarrow \overline{\mathbb{CP}^2} \setminus D4$

$= 0$

We now define k as follows:

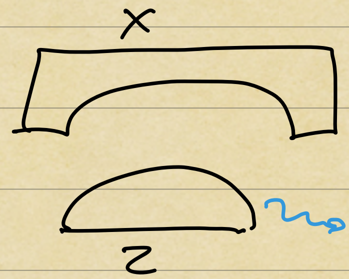
k : 

$\rightsquigarrow$

$G \cong$ 

$\to$ $\partial k + k \partial = m_2 + h_1 f_0 + f_2 h_0$

Key Obs: $m_Z =$



$\simeq \mathrm{Id}$

$\rightsquigarrow \overline{\mathbb{CP}^2} \setminus S^1 \times D^3$

To show this, first assume

$$\mathcal{M}_{G(S)}(Z) \xrightarrow{\cong} \mathcal{X} := \mathcal{X}(S^1\times S^2) \cong [-1,1]$$

Now, $\mathcal{M}(X) \times_{\mathcal{X}} \mathcal{M}_{G(S)}(Z) \cong \mathcal{M}(X) \times_{\mathcal{X}} \mathcal{M}(S^1\times D^3)$

for $\mathcal{M}(S^1\times D^3) \xrightarrow{\cong} \mathcal{X}$

And $\mathcal{M}(X) \times_{\mathcal{X}} \mathcal{M}(S^1\times D^3)$ cobordant to $\mathcal{M}([0,1]\times Y_0)$

for $X \cup S^1\times D^3 \cong [0,1]\times Y_0$